5.22 a. Let $G$ be $k$-connected, $e$ an edge of $G$. Prove that $G - e$ is $k - 1$-connected.

Let $S$ be a set of vertices, with $(G - e) - S$ either disconnected or equal to $K_1$.

If $e \notin E(G - S)$, then $(G - e) - S = (G - S) - e = G - S$, and so $G - S$ also is disconnected or equal to $K_1$.

Therefore $|S| \geq k > k - 1$.

Consider next if $e \in E(G - S)$. Then $G - S$ is not $K_1$, since a graph with an edge must have more than one vertex. If $G - S$ is disconnected, then $|S| \geq k > k - 1$. Suppose then that $G - S$ is connected. But $(G - e) - S$ is disconnected, and $(G - e) - S = (G - S) - e$, so $e$ is a cut edge of $G - S$. It $G - S$ equals $K_2$, then removing either of the vertices gives $K_1$. If $G - S$ has at least 3 vertices, then one of the vertices of the cut-edge $e$, call it $v$, must be a cut-vertex of $G - S$. Therefore removing $S \cup \{v\}$ from $G$ gives either $K_1$ or a disconnected graph. So $\left|S \cup \{v\}\right| \geq k$, that is $|S| \geq k - 1$.

We’ve now shown in all cases that the removal of at least $k - 1$ vertices from $G - e$ is needed to give either a disconnected graph or $K_1$. Therefore $G - e$ is $k - 1$-connected, as claimed.

5.23 b. If $G$ is a $k$-edge-connected graph, prove that $v + G$ is $k + 1$-edge-connected, where $v$ is a vertex not in $G$.

Let $T$ be a cut set of edges for $v + G$. Put $T_1 = \{ e \mid e \in T, v \in e \}$ and $T_2 = \{ e \mid e \in T, v \notin e \}$. Given vertices $a, b$ in $G$, then $\{v, a\}$ and $\{v, b\}$ are edges in $v + G$, and $a v b$ a path joining $a$ and $b$. Were $T_1 = \emptyset$, then $(v + G) - T$ would be connected, which is false. So $T_1 \neq \emptyset$, that is $|T| = |T_1| + |T_2| \geq 1 + |T_2|$. As $T_2$ disconnects $G$, we have $|T_2| \geq k$, so $|T| \geq 1 + k$, that is, $v + G$ is $k + 1$-edge-connected.

6.6 Let $G$ be a connected $r$-regular graph that is not eulerian. Prove that if $\overline{G}$ is connected, then $\overline{G}$ is eulerian.

Suppose $G$ has $n$ vertices. As $G$ is $r$-regular, $n$ must be even, say $n = 2m$. Now $G$ is connected, but not eulerian, so $r \equiv 1 \mod 2$. Also, $\overline{G}$ is regular of degree $2m - 1 - r$, and $2m - 1 - r \equiv 0 + 1 + 1 = 2 \equiv 0 \mod 2$.

Therefore, if $\overline{G}$ is connected, it must be eulerian, since each of its vertices has even degree.

6.12 Let $G$ be a 3-regular graph with 12 vertices, $H$ a 4-regular graph with 11 vertices.

a. If $v \in V(G)$ and $w \in V(H)$, then $\deg_{G + H}(v) = \deg_G(v) + \left|V(H)\right| = 3 + 11 = 14$ and $\deg_{G + H}(w) = \deg_H(w) + \left|V(G)\right| = 4 + 12 = 16$. As $G + H$ is connected, with every vertex of even degree, it is eulerian.

b. As $G + H$ has 23 vertices, with $\deg_{G + H}(v) = 14 \geq \frac{23}{2}$ and $\deg_{G + H}(w) = 16 \geq \frac{23}{2}$, we conclude, by Dirac’s theorem, that $G + H$ is hamiltonian.