Energy transfer between the shape and volume modes of a nonspherical gas bubble

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A model of a nonspherical gas bubble is developed in which the Rayleigh-Plesset equation is augmented with second order terms that back-couple the volume mode to a single shape mode. These additional terms in the Rayleigh-Plesset equation permit oscillation energy to be transferred back and forth between the shape and volume modes. The parametric stability of the shape mode is analyzed for a driven bubble, and it is seen that the bidirectional coupling yields an enhanced, albeit minor, stabilizing effect on the shape mode when compared with a model where the shape-volume coupling is unidirectional. It is also demonstrated how a pure shape distortion can excite significant volume pulsations when the volume mode is in 2:1 internal resonance with the shape mode.

I. INTRODUCTION

Instabilities which have been implicated in the excitation of nonspherical surface deformations of a gas bubble include the Rayleigh-Taylor instability,1–5 the Faraday (parametric) instability,6, 7 and the afterbounce instability.8, 9 A bubble will become Rayleigh-Taylor unstable due to a rapid acceleration of the bubble gas into the higher density liquid surrounding the bubble. The Rayleigh-Taylor instability acts on a very fast time scale and can destroy a bubble within a single cycle if conditions allow it to develop unabated. On the other hand, parametric instability is a cumulative effect for a driven bubble, requiring many oscillation cycles to build up. In the parametric instability, energy is transferred from the volume mode to the shape mode during each acoustic driving cycle and eventually overwhelms the bubble. Moreover, in some instances when the bubble expands with the driving pressure and collapses, there will be many smaller afterbounces of the bubble radius before the next driving cycle begins. In this situation, which is of particular interest for sonoluminescing bubbles, the parametric instability arises due to a resonance with the afterbounces.

Less widely appreciated than the surface instabilities is the fact that nonspherical distortion modes can excite volume pulsations via nonlinear mode coupling.10–15 Although typically a second order effect on the dynamics of the volume mode, resonant shape to volume coupling was proposed as a mechanism which could lead to a possibly significant natural source of underwater sound.14, 15 Experimental evidence of the excitation of the volume mode by the shape mode has been detected.16 This sort of nonlinear mode coupling has also been suggested to explain the “dancing bubble” phenomenon, where transfer of energy from surface and volume modes to the translation mode may explain the observed erratic drift of bubbles in liquids.17–19

In this article, we employ an energy balance approach to derive a model for a nonspherical gas bubble consisting of fully coupled ordinary differential equations that allow for energy to be transferred bidirectionally between the volume mode and a single shape mode (Sec. II). The equation for the volume mode is the well-known Rayleigh-Plesset equation, but augmented with second order
terms that capture the influence of the shape mode oscillations on the volume mode. The effect that the second order shape mode terms in the Rayleigh-Plesset equation have on the parametric stability of a bubble is examined in Sec. III, and their effect on the volume mode pulsations is examined in Sec. IV.

II. SHAPE-VOLUME MODEL OBTAINED FROM A LAGRANGIAN

We consider an isolated, nonspherical gas bubble immersed in an incompressible and slightly viscous fluid of infinite extent. A model for the motion of the bubble will be derived in the context of potential flow theory, including a correction for the slight viscosity of the fluid. Both mass and heat transfer are neglected in this work.

A. Velocity potential of an axisymmetric bubble

For simplicity, consider a bubble surface described in axisymmetric spherical coordinates by

\[ r = R(t)(1 + \varepsilon(t)P_n(\cos \theta)), \]

where \( R(t) \) corresponds to the spherical (or volume) pulsations of the bubble, and \( \varepsilon(t) \) governs the small amplitude oscillations of a single shape mode represented by the \( n \)th Legendre polynomial \( P_n(\cos \theta) \). The velocity potential of the fluid surrounding the bubble can be expressed in the form

\[
\Phi(r, \theta) = -\frac{1}{4\pi} \frac{\dot{V}(t)}{r} - \frac{E(t)}{r^{n+1}} P_n(\cos \theta) - \sum_{k=1}^{\infty} \frac{C_k(t)}{r^{k+1}} P_k(\cos \theta),
\]

where the volume of the bubble to \( O(\varepsilon^2) \) is

\[ V(t) = \frac{4}{3} \pi R^3 \left( 1 + \frac{3\varepsilon^2}{2n+1} \right). \]

On the bubble surface, the kinematic boundary condition requires that the normal component of the fluid velocity must equal the velocity of the bubble wall. If we express the bubble surface in level-set form \( F(r, \theta, t) = R(1 + \varepsilon P_n(\cos \theta)) - r = 0 \), then the kinematic boundary condition can be stated as

\[
\frac{\partial F}{\partial t} + \mathbf{u} \cdot \nabla F = 0 \quad \text{on} \quad F = 0,
\]

where \( \mathbf{u} = \nabla \Phi \). Substituting (1) into (2) and expanding for small \( \varepsilon \), we find that the coefficient \( E(t) \) in the velocity potential is

\[ E(t) = \frac{R^{n+1}}{n+1} (R^2 \dot{\varepsilon} + 3R\dot{R}\varepsilon). \]

The coefficients \( C_k(t) \), \( k \geq 1 \) in the velocity potential are determined at \( O(\varepsilon^2) \) from the kinematic boundary condition as

\[
C_k(t) = \left( 3(n+1)a_k - \frac{3}{n+1} d_k \right) \frac{R^{k+2} \dot{R} \dot{\varepsilon}^2}{k+1} + \left( (n+2)a_k - \frac{1}{n+1} d_k \right) \frac{R^{k+3} \ddot{R} \dot{\varepsilon}}{k+1},
\]

where

\[
a_k = \frac{2k+1}{2} \int_0^\pi P_n^2(\cos \theta) P_k(\cos \theta) \sin \theta \, d\theta,
\]

\[
d_k = \frac{2k+1}{2} \int_0^\pi \left( \frac{d}{d\theta} P_n(\cos \theta) \right)^2 P_k(\cos \theta) \sin \theta \, d\theta.
\]

A more general nonspherical bubble would have a surface shape consisting of a superposition of modes represented by all the spherical harmonics. The velocity potential we use in this work agrees with that given in Ceschia and Nabergoj when only a single mode is considered for the surface distortion.
B. Energy balance for a shape-volume bubble

The energy balance for an inviscid, incompressible fluid containing an immersed gas bubble can be expressed as a sum of kinetic and potential energies,

$$\Delta T_{\text{gas}} + \Delta U_{\text{gas}} + \Delta T_{\text{liquid}} + \Delta U_{\text{interface}} = \Delta W.$$ 

The energy balance equates the change in the total kinetic and potential energies in the system to the work done by the external pressure on the bubble as it changes its volume. Assuming the density of the gas inside the bubble is much smaller than the density of the liquid, $\rho$, outside the bubble, we can neglect the kinetic energy of the gas, $\Delta T_{\text{gas}}$. Therefore, the kinetic energy of the system resides in the liquid, and the potential energy of the system is in the compressible gas and the interface. At this point, viscous dissipation is not yet accounted for in the system, but that effect will be added to the model via the dissipation function in the Euler-Lagrange equations in Sec. II C.

If the uniform pressure of the gas inside the bubble is assumed to obey the equation of state

$$p_g = p_0g(V(0)/V(t))^\gamma,$$

then the potential energy of the bubble gas is given by

$$\Delta U_{\text{gas}} = - \int_{V(0)}^{V(t)} p_g(V') dV' = \begin{cases} -p_0g V(0) \ln \left( \frac{V(t)}{V(0)} \right), & \gamma = 1 \\ -\frac{p_0gV(0)}{\gamma-1} \left[ 1 - \left( \frac{V(t)}{V(0)} \right)^{\gamma-1} \right], & \gamma > 1. \end{cases}$$

The equilibrium bubble volume is $V(0) = (4/3)\pi R_0^3$ and $\gamma$ is the polytropic exponent of the gas.

The equilibrium pressure of the gas inside the bubble is $p_g = p_0^\gamma + 2\sigma/R_0$ with $p_0^\gamma$ being the ambient liquid pressure and $\sigma$ the surface tension of the liquid.

The kinetic energy of the liquid surrounding the bubble is

$$\Delta T_{\text{liquid}} = \int S \frac{\partial \Phi}{\partial \mathbf{n}} dS,$$

where $\Phi$ has the form (1), and the integral is evaluated over the surface of the bubble, $S$, with $\mathbf{n}$ the inward pointing unit normal vector to the bubble surface. To order $\varepsilon^2$, the kinetic energy of the liquid is calculated to be $\Delta T_{\text{liquid}} = T_{\text{liquid}}(t) - T_{\text{liquid}}(0)$, where

$$T_{\text{liquid}}(t) = 2\pi \rho R_0^3 \dot{R}^2 + \frac{2\pi \rho}{(2n+1)(n+1)} \left[ R^5 \dot{\varepsilon}^2 + 2(n+4)R \dot{\varepsilon} \ddot{\varepsilon} + (n+10)R^3 \dot{R}^2 \varepsilon^2 \right].$$

This expression for the kinetic energy agrees with that found in Ceschia and Nabergoj.20

The potential energy stored at the surface of the bubble is

$$\Delta U_{\text{interface}} = \int S \frac{\partial \Phi}{\partial \mathbf{n}} dS = \sigma(S(t) - S(0)),$$

where $\sigma$ is the surface tension of the liquid, and the surface area of the bubble is given at $O(\varepsilon^2)$ by

$$S(t) = 4\pi R^2 \left[ 1 + \frac{(n^2 + n + 2) \varepsilon^2}{2(2n+1)} \right].$$

The work done by the time-varying acoustic pressure is

$$\Delta W = -\int_{V(0)}^{V(t)} p^\infty(t) dV',$$

where $p^\infty(t) = p_0^\gamma - p_A \sin(\Omega t)$ represents the isotropic, externally applied pressure in the liquid.

For a slightly viscous fluid, the bubble will be surrounded by a thin boundary layer of vorticity,9 which we do not consider in our model. So, the rate at which energy is lost due to viscous dissipation can be expressed as

$$D = \frac{\mu}{2} \int_S \frac{\partial (\mathbf{u} \cdot \mathbf{n})}{\partial \mathbf{n}} dS.$$
which, to order $\epsilon^2$, yields
\[
D = 8\pi \mu R \ddot{R}^2 + \frac{4\pi \mu(n + 2)}{n + 1} R^3 \epsilon^2 + \frac{8\pi \mu(3n^2 + 10n + 4)}{(n + 1)(2n + 1)} R^2 \dot{R} \epsilon \dot{\epsilon} + \frac{12\pi \mu(5n + 2)}{(n + 1)(2n + 1)} R \ddot{R}^2 \epsilon^2, 
\]
where $\mu$ is the viscosity of the liquid. Other forms of energy dissipation for a gas bubble such as radiation damping and losses due to thermal effects will not be considered in this work.

C. Lagrangian for a shape-volume bubble

Construct the Lagrangian $L = T - U$, where $T = T_{\text{liquid}}$ and $U = U_{\text{gas}} + U_{\text{interface}}$. The Euler-Lagrange equations, with the dissipation function included, are
\[
\frac{d}{dt} \frac{\partial L}{\partial \dot{R}} - \frac{\partial L}{\partial R} = \frac{\partial W}{\partial R} - \frac{\partial D}{\partial R},
\]
\[
\frac{d}{dt} \frac{\partial L}{\partial \dot{\epsilon}} - \frac{\partial L}{\partial \epsilon} = \frac{\partial W}{\partial \epsilon} - \frac{\partial D}{\partial \epsilon}.
\]
The first Euler-Lagrange equation produces the classical Rayleigh-Plesset equation augmented by second order shape terms,
\[
R \ddot{R}(1 + N_1 \epsilon^2) + \frac{3}{2} R^2 (1 + N_1 \epsilon^2) = \left(\frac{p_g(V) - p_{\infty}(t)}{\rho}\right)(1 + N_2 \epsilon^2) - \frac{2\sigma}{\rho R} (1 + N_3 \epsilon^2)
\]
\[- \frac{4\mu \dot{R}}{R} (1 + N_4 \epsilon^2) - N_5 \frac{\mu \epsilon \dot{\epsilon}}{R} - N_6 R^2 \epsilon \dot{\epsilon}^2 - 2N_1 R \dot{R} \epsilon \dot{\epsilon} - N_7 R^2 \epsilon \dot{\epsilon},
\]
where
\[
N_1 = \frac{n + 10}{(2n + 1)(n + 1)}, \quad N_2 = \frac{3}{2n + 1}, \quad N_3 = \frac{n^2 + n + 2}{2(2n + 1)}, \quad N_4 = \frac{3(5n + 2)}{2(2n + 1)(n + 1)},
\]
\[
N_5 = \frac{2(3n^2 + 10n + 4)}{(2n + 1)(n + 1)}, \quad N_6 = \frac{2n + 3}{(2n + 1)(n + 1)}, \quad N_7 = \frac{n + 4}{(2n + 1)(n + 1)}.
\]
The second Euler-Lagrange equation produces the well-known shape mode equation
\[
\ddot{\epsilon} + A(t) \dot{\epsilon} + B(t) \epsilon = 0,
\]
where
\[
A(t) = \frac{5}{2} \frac{\dot{R}}{R} + 2(2n + 1)(n + 2) \frac{\mu}{\rho R^2}
\]
and
\[
B(t) = 3 \frac{R^2}{R^2} + (2 - n) \frac{\dot{R}}{R} + (n + 1)(n - 1)(n + 2) \frac{\sigma}{\rho R^3} + 6n(n + 2) \frac{\mu}{\rho R^3}.
\]
Equation (3) is the Rayleigh-Plesset equation plus $O(\epsilon^2)$ terms that account for the influence of the shape oscillations on the volume pulsation. Equation (4) governing the shape oscillations agrees exactly with the well-known result found in the literature.\(^7\) We note that we have used $r = R(t) [1 + \epsilon(t) P_n(\cos \theta)]$ to represent the bubble surface, whereas some authors express a nonspherical bubble surface as $r = R(t) + a_n(t) P_n(\cos \theta)$. Thus, to transform the shape mode equation (4) into the shape mode equation found in those references, one would make the change of variables $a_n = \epsilon R$.

We will refer to the nonspherical bubble model consisting of Eqs. (3) and (4) as the bidirectional shape-volume model, the two-way arrow as a reminder that the differential equations are fully coupled. The model in which the $O(\epsilon^2)$ terms are dropped from Eq. (3) will be referred to as the unidirectional shape-volume model, where the left arrow indicates that oscillation energy is only transferred one way, from the volume mode to the shape mode.
III. PARAMETRIC STABILITY OF THE SHAPE MODE

The surface of a pulsating gas bubble may be subjected to parametric and Rayleigh-Taylor instabilities. The Rayleigh-Taylor instability can rapidly initiate nonspherical surface distortions and possibly lead to bubble breakup within a few oscillation cycles. The parametric instability slowly causes growth of surface distortions by resonantly transferring oscillation energy from the acoustically driven volume mode to surface shape modes. Moreover, the transfer of energy from the volume mode to the shape modes can be resonant with respect to either the frequency of the driving sound field, or resonant with respect to “afterbounces” that occur for bubbles undergoing large acoustic forcing. Figure 1 depicts how the parametric instability can develop over time when oscillation energy is transferred from the volume mode to the shape mode.

When the acoustic forcing is small, we can examine the parametric stability of the shape mode using Floquet theory. For the moment, consider the unidirectional shape-volume model and let \( R(t) = R_0(1 + x(t)) \) with \( x(t) \) sufficiently small. Then the linearized Rayleigh-Plesset equation is given in nondimensional form by

\[
\ddot{x} + \beta \dot{x} + \Lambda^2 x = \alpha \sin \tau, \tag{5}
\]

where \( \tau = \frac{\omega}{\Omega} t \) and

\[
\beta = \frac{4\mu}{\Omega \rho R_0^2}, \quad \Lambda = \frac{\omega_0}{\Omega}, \quad \omega_0^2 = \frac{3\gamma p_\infty^\infty R_0^2}{\rho R_0^2} + \frac{2\sigma (3\gamma - 1)}{\rho R_0^3}, \quad \alpha = \frac{p_A}{\Omega^2 \rho R_0^2}.
\]

The long-time solution of (5) is given by

\[
R(t) = R_0[1 + b \sin(\tau + \psi)], \tag{6}
\]

where

\[
b = \frac{\alpha}{\sqrt{(\Lambda^2 - 1)^2 + \beta^2}} \quad \text{and} \quad \tan \psi = \frac{\beta}{1 - \Lambda^2}.
\]

We will assume that \( \beta \) is small enough so that the phase \( \psi \) can be neglected in (6).

Substituting (6) into the shape-mode equation (4) and neglecting terms of \( O(b^2) \) gives

\[
\ddot{\varepsilon} + [\zeta + 5b \cos \tau] \dot{\varepsilon} + [\lambda^2 + (n - 2 - 3\lambda^2)b \sin \tau] \varepsilon = 0, \tag{7}
\]

where again we have nondimensionalized by \( \tau = \Omega t \) and

\[
\lambda = \frac{\omega_n}{\Omega}, \quad \zeta = 2(2n + 1)(n + 2)\frac{\mu}{\rho \Omega R_0^2} \equiv O(b).
\]

![FIG. 1. Parametric instability of an \( n = 2 \) unidirectional-coupled shape-volume bubble in water. The equilibrium radius is \( R_0 = 65 \mu m \), and the volume mode is driven at \( \Omega/2\pi = 20.6 \) kHz, \( p_A = 0.15 \) bar.]
and the natural oscillation frequency of the \( n \)th shape mode is
\[
\omega_n^2 = \frac{(n+1)(n+2)(n-1)}{\rho R_0^3} \sigma.
\]

To proceed with the stability analysis, substitute the perturbation expansions
\[
\varepsilon(\tau) = \varepsilon_0(\tau) + b\varepsilon_1(\tau) + O(b^2),
\]
into Eq. (7), which yields
\[
\mathcal{O}(1) : \ddot{\varepsilon}_0 + \lambda_0^2 \varepsilon_0 = 0,
\]
\[
\mathcal{O}(b) : \ddot{\varepsilon}_1 + \lambda_0^2 \dot{\varepsilon}_1 = -(\zeta/b + 5 \cos \tau) \dot{\varepsilon}_0 - [\lambda_1^2 + (n - 2 - 3\lambda_0^2) \sin \tau] \varepsilon_0.
\]

The leading order solution for \( \varepsilon(\tau) \) is
\[
\varepsilon_0(\tau) = C \cos(\lambda_0 \tau) + D \sin(\lambda_0 \tau).
\]
As a specific example, we calculate the stability boundary emanating from \( \lambda_0 = 1/2 \) by substituting (12) into the right-hand side of Eq. (11)
\[
\ddot{\varepsilon}_1 + \frac{1}{4} \varepsilon_1 = - \left( \lambda_1^2 D + \left( \frac{n}{2} - \frac{1}{8} - \frac{\zeta}{2b} \right) C \right) \sin \frac{\tau}{2} - \left[ \lambda_1^2 C + \left( \frac{n}{2} - \frac{1}{8} + \frac{\zeta}{2b} \right) D \right] \cos \frac{\tau}{2} + \cdots.
\]

To eliminate the secular terms, we require that
\[
\lambda_1^2 C + \left( \frac{n}{2} - \frac{1}{8} + \frac{\zeta}{2b} \right) D = 0,
\]
\[
\left( \frac{n}{2} - \frac{1}{8} - \frac{\zeta}{2b} \right) C + \lambda_1^2 D = 0.
\]

In order that these equations are satisfied for arbitrary initial conditions, the determinant of the system must be zero, which yields
\[
\lambda_1^4 = \left( \frac{n}{2} - \frac{1}{8} \right)^2 - \frac{\zeta^2}{4b^2}.
\]

Upon recalling expansion (9), we see that in the \( \lambda - b \) parameter plane there is a small neighborhood of \( \lambda = 1/2 \) and \( b = 0 \) where the set of parameters corresponding to stable shape oscillations is
\[
b < \pm \left( \frac{8}{4n-1} \right) \left( \lambda_1^2 - \frac{1}{4} \right)^2 + \frac{\zeta^2}{4} \right)^{1/2}.
\]

For an inviscid fluid, \( \zeta = 0 \), the stability threshold becomes
\[
p_A < \frac{8\rho R_0^2 \omega_0^2}{4n-1} \left( 1 - \frac{\Omega^2}{\omega_0^2} \right) \left( \frac{\omega_n^2}{\Omega^2} - \frac{1}{4} \right).
\]

In Figure 2, the parametric stability diagram for unidirectional-coupled shape←volume bubbles near the \( \omega_2: \Omega = 1:2 \) resonance is computed numerically and compared to the analytical stability threshold given by (13). For the numerical simulations, the material parameters are \( \rho = 998 \text{ kg/m}^3 \), \( \sigma = 0.0725 \text{ N/m}, p_\infty = 1 \text{ bar}, \) and \( \gamma = 1 \). With the driving frequency set at \( \Omega/2\pi = 20.6 \text{ kHz} \), each \( (R_0, p_A) \) pair was checked for shape stability in the following manner. Given a bubble initially at rest with \( R(0) = R_0 \) and \( \varepsilon(0) = 0.01 \), the unidirectionally-coupled shape←volume equations of motion were integrated for 300 cycles of the acoustic forcing. If, after 300 driving cycles, the shape mode
was seen to be stable to the initially small amplitude surface distortion, then the \((R_0, p_A)\) point was colored black, otherwise it was colored white.

Observe in Figure 2 that the stability boundary (Arnold tongue) emanates from approximately \(R_0 \approx 59\mu m\), which corresponds to the shape resonance \(\omega_2: \Omega = 1:2\). Also note that there is a dip in the stability boundary of Figure 2 at roughly \(R_0 \approx 67\mu m\), which corresponds to the volume resonance, \(\omega_0: \Omega = 2:1\). Figure 3 (bottom) shows the resonance curves for both the shape and volume modes, alongside the parametric stability diagram for a wider range of bubble sizes, Figure 3 (top). Shape stability diagrams such as Figures 2 and 3, as well as positional and diffusional stability criteria, are especially important in understanding the parameter regimes of stable single-bubble sonoluminescence.\(^9\), \(^{21}\), \(^{22}\)

The parametric stability diagram for the bidirectional shape↔volume model is shown in Figure 4 (top). Comparing Figures 3 (top) and 4 (top) we observe that the coupling of \(O(\varepsilon^2)\) shape mode terms into the Rayleigh-Plesset equation has a minor stabilizing effect on the parametric instability of the bubble surface. In Figure 4 (bottom), the black regions indicate stability for bidirectional-coupling, but not for unidirectional-coupling.

**IV. EXCITATION OF THE VOLUME MODE BY THE SHAPE MODE**

In order to better understand the natural mechanisms of underwater sound generation near the sea surface, such as from a breaking wave or a raindrop, Longuet-Higgins\(^{14,15}\) proposed that
pulsating nonspherical air pockets entrained near the surface of the ocean can be a significant source of underwater sound. Longuet-Higgins demonstrated that the nonspherical distortion modes of an oscillating gas bubble, whose pressure fields decay radially like $r^{-(n+1)}$, can give rise to surprisingly intense monopole radiation of sound ($n = 0$) due to nonlinear coupling of the shape modes with the volume mode. By analyzing the bidirectionally-coupled shape↔volume model developed in this paper, we can examine how shape oscillations can indeed excite the volume mode.

Consider a freely oscillating, undamped nonspherical bubble with initial size $R(0) = R_0$ and $\varepsilon(0) = \varepsilon_0$. With these initial conditions, if the unidirectional shape↔volume model is used to simulate the oscillations, then the volume mode will remain at equilibrium for all time, since its governing equation is the Rayleigh-Plesset equation with no $O(\varepsilon^2)$ terms coupled to it. On the other hand, if the bidirectional shape↔volume model is employed, then the influence of the second order shape mode terms on the Rayleigh-Plesset equation will initiate a small amplitude volume mode pulsation. If we also imagine that these induced volume oscillations are small enough, then for a sufficiently short time they will not influence the shape oscillations too much. Therefore, we can linearize the coupled Rayleigh-Plesset equation using the shape oscillations as the driving force. Let $R(t) = R_0(1 + \varepsilon(t))$ and expand Eq. (3) by neglecting terms of size $O(\varepsilon^2, \varepsilon^3)$ but retaining terms of size $O(\varepsilon)$,

$$\ddot{x} + \omega_0^2 x = -C\varepsilon^2 - N_6\varepsilon^2 - N_7\varepsilon\varepsilon.$$

(15)
The coefficients \( N_6 \) and \( N_7 \) are given in Sec. II C, and \( C = N_2 p_0^\infty/\rho R_0^2 + 2\pi N_3/\rho R_0^3 \). Substituting shape oscillations of the form \( \varepsilon(t) = \varepsilon_0 \cos(\omega_0 t) \) into (15) yields

\[
\ddot{x} + \omega_0^2 x = A + B \cos(2\omega_0 t),
\]

where \( A = -\varepsilon_0^2 (C/2 + Q_1 \omega_0^2) \) and \( B = -\varepsilon_0^2 (C/2 - Q_2 \omega_0^2) \) with \( Q_1 = (n - 1)/(2(n + 1)(2n + 1)) \) and \( Q_2 = (3n + 7)/(2(n + 1)(2n + 1)) \).

In agreement with the conclusion of Longuet-Higgins,\(^{14,15}\) it is plain to see in Eq. (16) that the volume mode will be in resonance with the shape mode when \( \omega_0 = 2\omega_n \). Away from this resonance, the volume mode oscillations are given by \( R(t) = R_0(1 + x(t)) \) with

\[
x(t) = F + G \cos(\omega_0 t) + H \cos(2\omega_0 t),
\]

where

\[
F = -\varepsilon_0^2 (C + 2Q_1 \omega_0^2), \quad H = \varepsilon_0^2 (2Q_2 \omega_0^2 - C), \quad G = \varepsilon_0^2 [C(\omega_0^2 - 2\omega_n^2) - \omega_n^2(N_7\omega_0^2 + 4Q_1\omega_n^2)]/\omega_0^2(\omega_0^2 - 4\omega_n^2).
\]

Note that the magnitude of the excited volume pulsation is proportional to the square of the shape mode amplitude. In Figure 5, we plot \( R(t) = R_0(1 + x(t)) \) with \( x(t) \) given by (17) along with the numerical solution of the bidirectional-coupled shape↔volume equations. In the figure we use
FIG. 5. The volume mode (top), initially at rest in this simulation at $R_0 = 50 \mu m$, starts to pulsate due to an $n = 2$ shape mode distortion (bottom). The linear solution (dashed curve) for a freely oscillating, undamped shape-volume bubble away from 2:1 resonance is plotted along with the numerical solution (solid curve) of the bidirectional shape-volume model.

$n = 2$ and $R_0 = 50 \mu m$, for which the breathing mode frequency is $\omega_0 = 55.7$ kHz and the shape mode frequency is $\omega_2 = 13.3$ kHz. Away from resonance, the linear solution provides a rough approximation to the dynamics of the excited volume mode for sufficiently short time. The material parameters used in the simulation are the same as those given in Sec. III.

In Figure 6 we plot the numerical solution of the bidirectional shape↔volume equations for $n = 2$ and $R_0 = 10 \mu m$, which is very near the $\omega_0 = 2\omega_n$ resonance. For this case, the breathing mode frequency is $\omega_0 = 289$ kHz and the shape mode frequency is $\omega_2 = 148$ kHz. We observe in Figure 6 that the resonant transfer of oscillation energy from the shape mode to the volume mode has a fairly significant effect on the bubble dynamics. We keep in mind though that the bubble model is valid for small surface distortions, and therefore expect the model will become less accurate as $\epsilon$ nears $\mathcal{O}(1)$ values.

FIG. 6. The volume mode, initially at rest in this simulation at $R_0 = 10 \mu m$, becomes excited by $n = 2$ resonant shape oscillations, $\omega_0 = 2\omega_n$. The numerical solution of the bidirectional-coupled shape↔volume model is computed for a freely oscillating, undamped bubble.
V. CONCLUSIONS

In this article, a Lagrangian was constructed for a nonspherical gas bubble immersed in an incompressible and slightly viscous liquid. The Euler-Lagrange equations were employed to obtain a fully coupled bubble model consisting of the well-known shape mode equation, and the Rayleigh-Plesset equation which is augmented with second order shape mode terms. The second order shape mode terms are those arising solely from the influence of a single nth shape mode. For simplicity of the model, all other second order shape mode terms were neglected, even though for a real bubble they too would certainly affect the dynamics of the volume mode. A parametric stability threshold for a driven bubble was derived near the resonance $\Omega = 2\omega_2$. Coupling the second order shape mode terms into the Rayleigh-Plesset equation yields an enhanced, albeit minor, stabilizing influence on the parametric surface instability compared to a model where they are not included. The stability of volume pulsations to energy transfer from free, undamped oscillations of a shape mode was also examined. It was shown how the volume mode can be excited by shape mode oscillations near the resonance condition, $\omega_{th} = 2\omega_n$, which is in agreement with the conclusion of Longuet-Higgins. Finally, we remark that by tracking the motion of specific points on the bubble surface, such as the poles and equator, it is possible to develop a collocation model of a nonspherical bubble which also has bidirectional transfer of energy.23

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