Geometry of Generalized Complex Numbers

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Alternative definitions of the imaginary unit i other than $i^2 = -1$ can give rise to interesting and useful complex number systems. The 16th-century Italian mathematicians G. Cardan (1501–1576) and R. Bombelli (1526–1572) are thought to be among the first to utilize the complex numbers we know today by calculating with a quantity whose square is -1. Since then, various people have modified the original definition of the product of complex numbers. The English geometer W. Clifford (1845–1879) developed the “double” complex numbers by requiring that $i^2 = 1$. Clifford’s application of double numbers to mechanics has been supplemented by applications to noneuclidean geometries. The German geometer E. Study (1862–1930) added still another variant to the collection of complex products. The “dual” numbers arose from the convention that $i^2 = 0$ [**11**]. Well known in kinematics is the use of dual number methods for the analysis of spatial mechanisms, robotic control, and virtual reality [**4, 5, 10**].

The ordinary, dual, and double numbers are particular members of a two-parameter family of complex number systems often called binary numbers or generalized complex numbers, which are two-component numbers of the form

$$z = x + iy \quad (x, y \in \mathbb{R}) \quad \text{where} \quad i^2 = iq + p \quad (q, p \in \mathbb{R}).$$

It can be shown that generalized complex number systems are isomorphic (as rings) to the ordinary, dual, and double complex numbers when $p + q^2/4$ is negative, zero, and positive, respectively (FIGURE 1) [**11**].

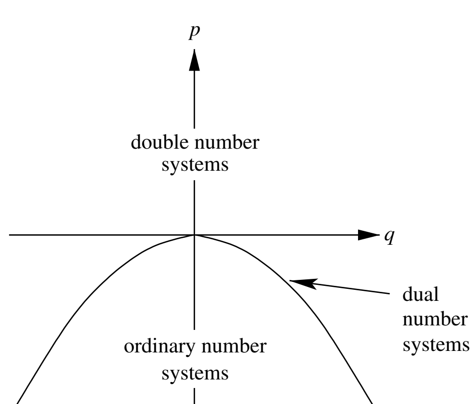


Figure 1 Generalized complex numbers are isomorphic (as rings) to the ordinary, dual, and double numbers.

In this article we study the geometry of a one-parameter family of generalized complex number systems in which $i^2 = p$, so that $q = 0$ and $-\infty < p < \infty$. Those who know the geometries of Laguerre and Minkowski will recognize that they arise naturally from generalized complex planes. Moreover, interrelations among the various complex products become obvious when the story of these planes unfolds.

Generalized complex multiplication

In what follows, we will let i denote a formal quantity, subject to the relation $i^2 = p$. Let $\mathbb{C}_p$ denote the system of numbers

$$\mathbb{C}_p = \{x + iy : x, y \in \mathbb{R}, \quad i^2 = p\}.$$

Addition and subtraction in this *p-complex plane* are defined, as usual, componentwise. Multiplication is also as we would expect, distributing multiplication over addition and using $i^2 = p$. Still, it will be helpful later on to introduce specific notation for this *p-multiplication*. So, for $z_1, z_2 \in \mathbb{C}_p$, we denote the product by

$$M^p(z_1, z_2) = (x_1x_2 + py_1y_2) + i(x_1y_2 + x_2y_1).$$

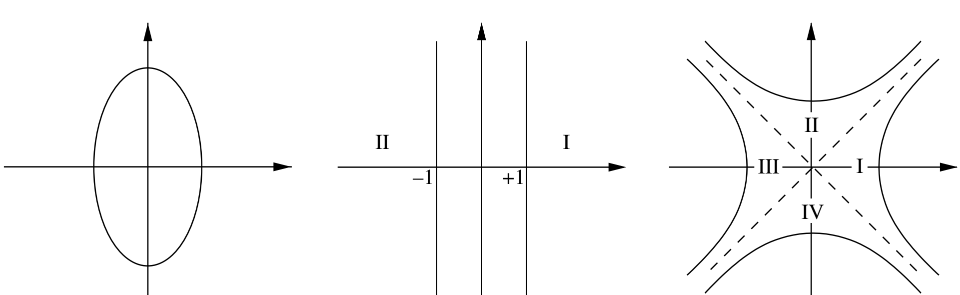
This definition yields the ordinary, Study, and Clifford products as p is equal to -1, 0, and 1.

Ordinary product: $(x_1 + iy_1)(x_2 + iy_2) = (x_1x_2 - y_1y_2) + i(x_1y_2 + y_1x_2)$
Study product: $(x_1 + iy_1)(x_2 + iy_2) = (x_1x_2) + i(x_1y_2 + y_1x_2)$
Clifford product: $(x_1 + iy_1)(x_2 + iy_2) = (x_1x_2 + y_1y_2) + i(x_1y_2 + y_1x_2)$

We note that $\mathbb{C}_p$, under addition and p-multiplication, is a field only for $p < 0$. The *p-magnitude* of a generalized complex number $z = x + iy \in \mathbb{C}_p$ is defined to by the nonnegative real number

$$\|z\|_p = \sqrt{|M^p(z, \overline{z})|} = \sqrt{|x^2 - py^2|}$$

where an overbar denotes the usual complex conjugation.



$p < 0$ $p = 0$ $p > 0$

Figure 2 Unit circles in $\mathbb{C}_p$

Unit "circles" are defined by requiring $\|z\|_p = 1$ as in FIGURE 2. When $p < 0$ we obtain unit ellipses of the form $x^2 + |p|y^2 = 1$, and refer to $\mathbb{C}_p$ $(p < 0)$ as an elliptical complex number system. In the special case $p = -1$, the p-complex plane corresponds to the Euclidean plane. For $\mathbb{C}_0$, where $\|z\|_0^2 = x^2$, the unit circle is the set of z where $x = \pm 1$. The space $\mathbb{C}_0$ is the parabolic complex number system whose p-complex plane corresponds to the Laguerre plane. The parabolic complex plane is naturally divided in half by the imaginary axis. The right-half plane of $\mathbb{C}_0$ will be referred to as branch I and the left half-plane branch II. Unit circles in $\mathbb{C}_p$ $(p > 0)$ are hyperbolas of the form $|x^2 - py^2| = 1$ whose asymptotes are $y = \pm x/\sqrt{p}$ (dashed lines in FIGURE 2). The spaces $\mathbb{C}_p$ $(p > 0)$ are referred to as hyperbolic complex number systems. For the special case $p = 1$, the p-complex plane is the well-known Minkowski plane. The asymptotes of the unit circles naturally separate the hyperbolic complex planes into four regions labeled branches I, II, III, and IV as shown in FIGURE 2.

Generalized trigonometry

Much of the geometrical insight into the ordinary complex plane is facilitated by the trigonometric form of a complex number. The same is true for generalized complex planes. Therefore, we now examine a trigonometry suitable for computations with generalized complex numbers.

Measures of angles The generalized complex number, $z = x + iy$, determines a ray $\overrightarrow{OT}$ as shown in FIGURE 3. Let the point N be the intersection of the ray $\overrightarrow{OT}$ and the unit circle in $\mathbb{C}_p$ (for now, suppose that z lies in the first hyperbolic branch). The p-argument of z, θ_p, is defined to be twice the Euclidean area of the shaded sector OMN determined by the arc MN and the radii $\overline{OM}$ and $\overline{ON}$. (The meanings of the words *sector*, *arc*, and *radius* should be clear from the picture.) Define the ratio $\sigma \equiv y/x$; then the geometric definition of angular measure yields formulae involving familiar inverse tangent functions:

$$\theta_p = \begin{cases} \frac{1}{\sqrt{|p|}} \tan^{-1}(\sigma\sqrt{|p|}), & p < 0 \\ \sigma, & p = 0 \\ \frac{1}{\sqrt{p}} \tanh^{-1}(\sigma\sqrt{p}), & p > 0 \text{ (branch I, III).} \end{cases}$$

The various factors of $\sqrt{p}$ simply account for the scaling of the unit ellipses and hyperbolas. Observe that angular measure can also be expressed succinctly as a power series:

$$\theta_p = \sum_{n=0}^{\infty} \frac{p^n}{2n+1} \sigma^{2n+1}, \qquad |\sigma|\sqrt{|p|} < 1.$$

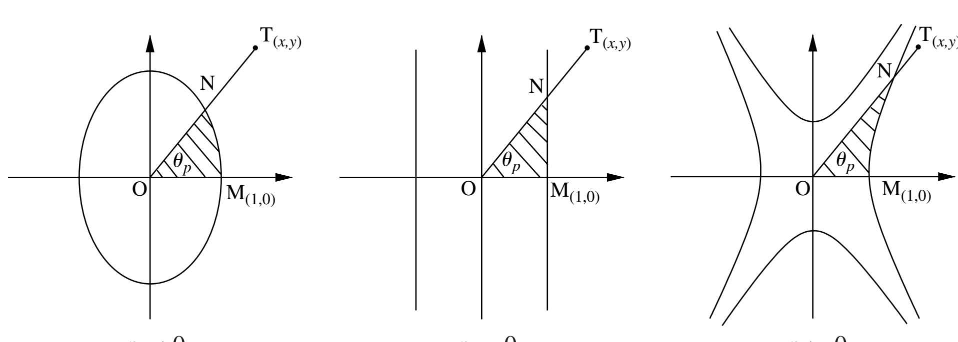


Figure 3 Elliptic, parabolic, and hyperbolic angles

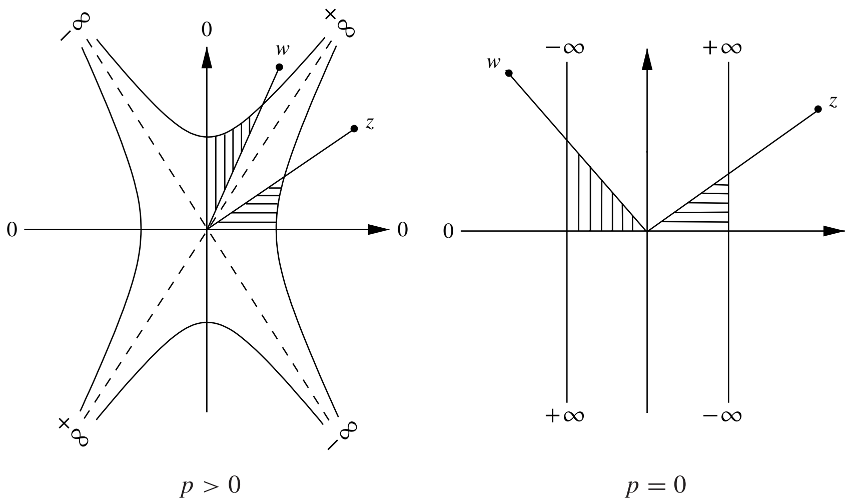


Figure 4 Angular measure extended to the whole hyperbolic and parabolic complex planes

The extension of angular measure throughout the entire parabolic and hyperbolic complex planes is suggested by FIGURE 4 and consists of some simple bookkeeping. Within each of the four branches of the hyperbolic complex plane (left FIGURE 4), angular measure is determined with respect to the half-axis that lies within the particular branch, and θ_p varies from $-\infty$ to $+\infty$ in the manner labeled on the asymptotes. Thus, for a hyperbolic complex number in branch II or IV, the angular measure is given by $\theta_p = (1/\sqrt{p})\tanh^{-1}[1/(\sigma\sqrt{p})] = (1/\sqrt{p})\coth^{-1}[\sigma\sqrt{p}]$. For example, in $\mathbb{C}_3$ the argument of $w = 2 + 5i$ is $\theta_3 = (1/\sqrt{3})\tanh^{-1}[2/(5\sqrt{3})] \approx 0.1358$ which is measured from the positive imaginary axis.

In both of the branches of the parabolic complex plane (right FIGURE 4), angular measure is given by $\theta_p = \sigma = y/x$. When the real part of a parabolic complex number is negative, then its angular measure is referenced with respect to the negative part of the real axis and the unit circle in $\mathbb{C}_0$. Hence, the orientation of angles in branch II is opposite that in branch I, as indicated in the figure.

Trigonometric functions From the point N on the unit circle in $\mathbb{C}_p$ drop the perpendicular $\overline{\text{NP}}$ to the radius $\overline{\text{OM}}$ (FIGURE 5). At the point M draw a line tangent to the unit circle. Let Q be the point of intersection of the tangent and the line through $\overline{\text{ON}}$. The

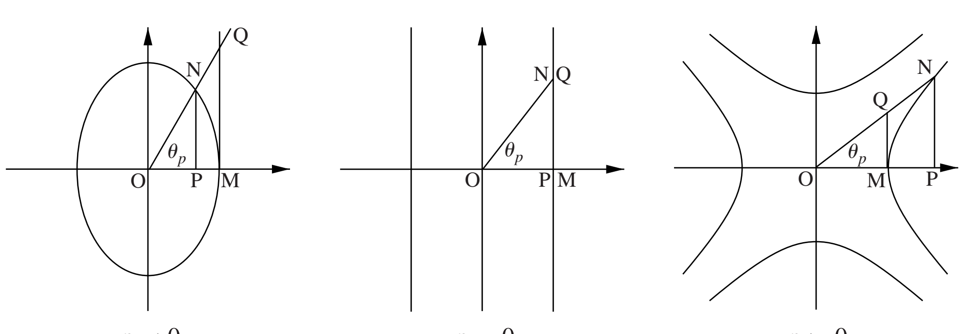


Figure 5 Geometric definitions of cosp, sinp, and tanp

lengths of the segments $\overline{\mathrm{OP}}$, $\overline{\mathrm{NP}}$, and $\overline{\mathrm{QM}}$ are defined to be the p-cosine (cosp), p-sine (sinp), and p-tangent (tanp), respectively. These geometric definitions give familiar expressions for the p-trigonometric functions:

$$\operatorname{cosp}\theta_p = \begin{cases} \cos(\theta_p\sqrt{|p|}), & p<0 \\ 1, & p=0 \text{ (branch I)} \\ \cosh(\theta_p\sqrt{p}), & p>0 \text{ (branch I)} \end{cases}$$

and

$$\operatorname{sinp}\theta_p = \begin{cases} \frac{1}{\sqrt{|p|}}\sin(\theta_p\sqrt{|p|}), & p<0 \\ \theta_p, & p=0 \text{ (branch I)} \\ \frac{1}{\sqrt{p}}\sinh(\theta_p\sqrt{p}), & p>0 \text{ (branch I).} \end{cases}$$

From the proportion QM/OM = NP/OP, we see that

$$\operatorname{tanp}\theta_p = \frac{\operatorname{sinp}\theta_p}{\operatorname{cosp}\theta_p}.$$

When $p=-1$ we find that the definitions reduce to the traditional circular trigonometric functions. Moreover, when $p=1$ the familiar hyperbolic functions are recovered.

The parabolic and hyperbolic trigonometric functions on the other branches of their respective complex planes can be naturally defined in terms of the trigonometric functions on branch I. In the parabolic complex plane, define $\operatorname{cosp}_{\mathrm{II}}\theta_p = -\operatorname{cosp}_{\mathrm{I}}\theta_p$ and $\operatorname{sinp}_{\mathrm{II}}\theta_p = -\operatorname{sinp}_{\mathrm{I}}\theta_p$, where the subscripts are a convenient way to keep track of branches. In the hyperbolic complex planes, let

$$\operatorname{cosp}_{\mathrm{II}}\theta_p = \frac{i}{\sqrt{p}}\operatorname{cosp}_{\mathrm{I}}\theta_p, \quad \operatorname{cosp}_{\mathrm{III}}\theta_p = -\operatorname{cosp}_{\mathrm{I}}\theta_p, \quad \operatorname{cosp}_{\mathrm{IV}}\theta_p = -\frac{i}{\sqrt{p}}\operatorname{cosp}_{\mathrm{I}}\theta_p$$

and

$$\operatorname{sinp}_{\mathrm{II}}\theta_p = \frac{\sqrt{p}}{i}\operatorname{sinp}_{\mathrm{I}}\theta_p, \quad \operatorname{sinp}_{\mathrm{III}}\theta_p = -\operatorname{sinp}_{\mathrm{I}}\theta_p, \quad \operatorname{sinp}_{\mathrm{IV}}\theta_p = -\frac{\sqrt{p}}{i}\operatorname{sinp}_{\mathrm{I}}\theta_p.$$

The Maclaurin expansions for cosp and sinp (branch I) are given by

$$\operatorname{cosp}\theta_p = \sum_{n=0}^{\infty}\frac{p^n}{(2n)!}\theta_p^{2n}$$

and

$$\operatorname{sinp}\theta_p = \sum_{n=0}^{\infty}\frac{p^n}{(2n+1)!}\theta_p^{2n+1}.$$

A generalized Euler's formula is obtained by comparing these Maclaurin series with the formal power series expansion for $e^{i\theta_p}$, recalling that $i^2=p$:

$$e^{i\theta_p} = \operatorname{cosp}\theta_p + i\ \operatorname{sinp}\theta_p.$$

Trigonometric identities The identity $|\operatorname{cosp}^2\theta_p - p\ \operatorname{sinp}^2\theta_p| = 1$ is evident, since $|x^2-py^2|=1$ is the form of a unit circle in $\mathbb{C}_p$. The next candidates for generalization are the addition laws for cosp and sinp. Let θ_p and ϕ_p be angular measures (in branch I when $p = 0$ or $p > 0$). Then

$$\operatorname{cosp}(\theta_p \pm \phi_p) = \operatorname{cosp}\theta_p \operatorname{cosp}\phi_p \pm p \operatorname{sinp}\theta_p \operatorname{sinp}\phi_p$$
$$\operatorname{sinp}(\theta_p \pm \phi_p) = \operatorname{sinp}\theta_p \operatorname{cosp}\phi_p \pm \operatorname{cosp}\theta_p \operatorname{sinp}\phi_p.$$

Demonstrating these is straightforward, since the formulas for cosp and sinp reduce, in each case, to situations where addition laws are known.

We also observe that when $p < 0$ the p-trigonometric functions are periodic with period $2\pi/\sqrt{|p|}$. In particular, let θ_p be an angular measure with $p < 0$ and $k = 0, 1, 2, 3, \ldots$, then

$$\operatorname{cosp}(\theta_p + 2k\pi/\sqrt{|p|}) = \operatorname{cosp}\theta_p$$
$$\operatorname{sinp}(\theta_p + 2k\pi/\sqrt{|p|}) = \operatorname{sinp}\theta_p.$$

Interpretation of generalized complex multiplication

The trigonometric forms of the real and imaginary parts of $z = x + iy$ in $\mathbb{C}_p$ are

$$x = r_p \operatorname{cosp}\theta_p$$
$$y = r_p \operatorname{sinp}\theta_p,$$

where $r_p = \|z\|_p$ is the p-magnitude of z, and θ_p is the p-argument of z. Therefore, the trigonometric form of a generalized complex number is

$$z = x + iy = r_p(\operatorname{cosp}\theta_p + i \operatorname{sinp}\theta_p).$$

The geometric significance of p-multiplication now becomes clear. Suppose we have two complex numbers in $\mathbb{C}_p$, for example $z = \|z\|_p(\operatorname{cosp}\theta_p + i \operatorname{sinp}\theta_p)$ and $w = \|w\|_p(\operatorname{cosp}\phi_p + i \operatorname{sinp}\phi_p)$. Using the definition of p-multiplication and then recalling the addition laws for cosp and sinp, we obtain

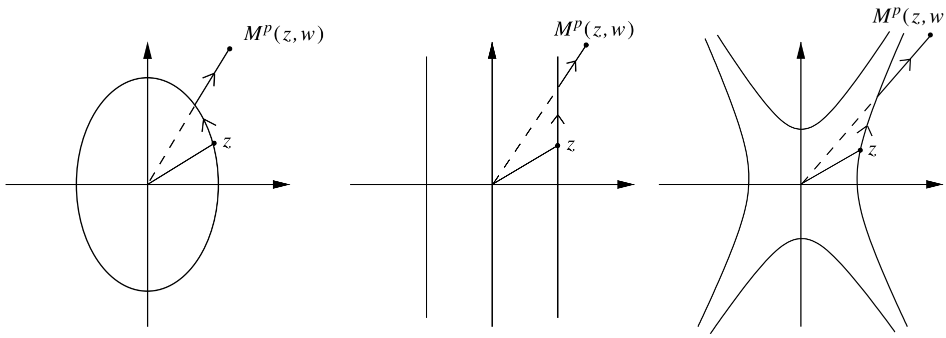
$$M^p(z, w) = \|z\|_p\|w\|_p\big(\operatorname{cosp}(\theta_p + \phi_p) + i \operatorname{sinp}(\theta_p + \phi_p)\big).$$

Hence the p-length of the product is the product of the p-lengths and the p-argument of the product is the sum of the p-arguments. Therefore, the product of two generalized complex numbers can be obtained via rotation and amplification, and it should be emphasized that the rotation is along a generalized circle in $\mathbb{C}_p$. More specifically, suppose we wish to multiply z with w as in FIGURE 6. The product $M^p(z, w)$ is derived geometrically by rotating z through an angle $\phi_p = \arg w$ along the generalized circle of radius $\|z\|_p$, and then expanding by a factor of $\|w\|_p$.

In FIGURE 7 we present pictorially a few concrete examples of the geometry of generalized complex multiplication. In each plot, the two complex numbers labeled with circles are being multiplied to produce the third complex number marked with an asterisk. The unit circles are shown for reference.

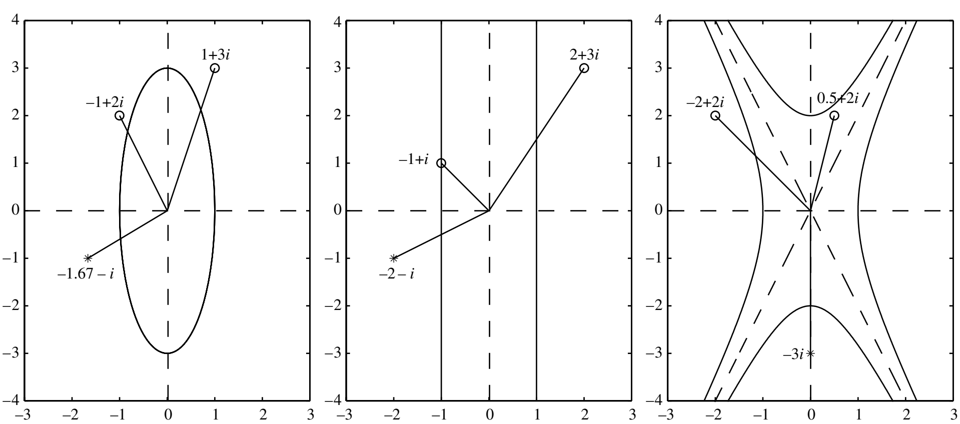
For complex products in the hyperbolic and parabolic complex planes, evaluating cosp and sinp requires keeping track of the branch into which the product falls. To see an example of how this can be done, we'll examine the multiplication of the two dual numbers in the $p = 0$ case of FIGURE 7. In that case, the definition of p-multiplication gives,

$$M^0(2 + 3i, -1 + i) = -2 - i.$$



$p < 0$ $p = 0$ $p > 0$

Figure 6 Multiplication is accomplished by rotation and amplification



$p = -1/9$ $p = 0$ $p = 1/4$

Figure 7 Geometric illustration of generalized complex multiplication

Alternatively, we can multiply the trigonometric forms of the numbers. The modulus of $2 + 3i$ in $\mathbb{C}_0$ is $\|2 + 3i\|_0 = 2$ and the argument is $\theta_p = 3/2$, which by definition of θ_p is twice the area of the triangle bounded by the real axis, the unit circle, and the ray connecting the origin to $2 + 3i$. So, the trigonometric form is $2 + 3i = 2(\operatorname{cosp}_{\mathrm{I}} \frac{3}{2} + i \operatorname{sinp}_{\mathrm{I}} \frac{3}{2})$. Similarly, $-1 + i = 1(\operatorname{cosp}_{\mathrm{II}}(-1) + i \operatorname{sinp}_{\mathrm{II}}(-1))$, which we rewrite as $(-\operatorname{cosp}_{\mathrm{I}}(-1) - i \operatorname{sinp}_{\mathrm{I}}(-1))$. We leave it to the reader to verify that multiplying the numbers in these yields $-2 - i$ as the product.

Generalized rotations and special relativity As seen in the previous section, the generalized complex product typically involves both an expansion (or contraction) and a generalized rotation. In the specific case where $\|w\|_p = 1$, the generalized complex product, $M^p(z, w)$, represents a pure rotation of z in $\mathbb{C}_p$. A pure rotation in $\mathbb{C}_p$ can be thought of as motion of the point z restricted to the generalized circle with radius $\|z\|_p$.

Generalized rotations can be applied to the theory of special relativity. In two-dimensional special relativity, an event that occurs at time t and at a space coordinate x is denoted by the spacetime point (t, x). Consider the generalized complex number $z = t + ix$ to be the spacetime coordinate of an event in $\mathbb{C}_p$, where $p = 1/c^2$ ($c \equiv$ speed of light). Let V represent the velocity of a coordinate frame (t', x') in uniform motion with respect to the inertial coordinate frame (t, x) of the event. If we now let

$w = 1 - iV$, then the pure rotation represented by the product

$$M^p\left(z, \frac{w}{\|w\|_p}\right) = \left[\frac{t - Vx/c^2}{\sqrt{1 - V^2/c^2}}\right] + i\left[\frac{x - Vt}{\sqrt{1 - V^2/c^2}}\right] = t' + ix'$$

yields the Lorentz coordinate transformations of two-dimensional special relativity. Hence, the Lorentz transformations of two-dimensional special relativity are simply rotations in the hyperbolic complex plane. In fact, if a velocity parameter, ϕ_p, is defined by $\operatorname{tanp}\phi_p = -V$, then the Lorentz transformation can be succinctly expressed as multiplication of $z = t + ix$ by $e^{i\phi_p}$. An article by Fjelstad [**6**] further explores the connection of hyperbolic complex numbers to special relativity.

Powers and roots of generalized complex numbers

Generalized De Moivre formulas allow us to compute powers and roots of complex numbers in $\mathbb{C}_p$.

THEOREM 1. (POWERS OF GENERALIZED COMPLEX NUMBERS) *For $z \in \mathbb{C}_p$ and n a positive integer,*

$$z^n = \left[r_p(\operatorname{cosp}\theta_p + i\ \operatorname{sinp}\theta_p)\right]^n = r_p^n\big(\operatorname{cosp}(n\theta_p) + i\ \operatorname{sinp}(n\theta_p)\big).$$

The proof is left to the reader, as it follows easily by induction, using the laws for p-multiplication and addition.

We will state two theorems concerning the computation of nth roots of complex numbers. The first theorem applies to complex numbers in $\mathbb{C}_p$ ($p < 0$) and the second theorem covers the cases when $p > 0$ and $p = 0$. For $p < 0$, the trigonometric functions are $2\pi/\sqrt{|p|}$ -periodic, leading to the following theorem on the extraction of nth roots of elliptical complex numbers.

THEOREM 2. (ROOTS OF ELLIPTICAL COMPLEX NUMBERS) *For z in $\mathbb{C}_p(p < 0)$ and n a positive integer,*

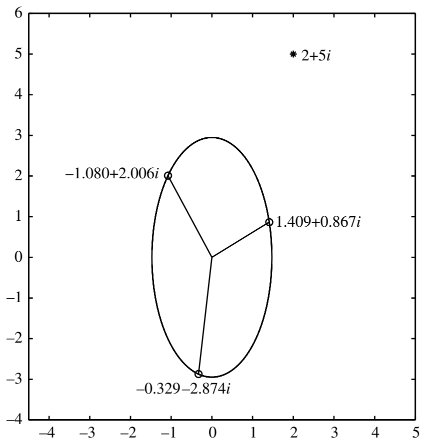
$$\begin{aligned} z^{\frac{1}{n}} &= \left[r_p(\operatorname{cosp}\theta_p + i\ \operatorname{sinp}\theta_p)\right]^{\frac{1}{n}} \\ &= r_p^{\frac{1}{n}}\left(\operatorname{cosp}\left(\frac{\theta_p + 2k\pi/\sqrt{|p|}}{n}\right) + i\ \operatorname{sinp}\left(\frac{\theta_p + 2k\pi/\sqrt{|p|}}{n}\right)\right), \end{aligned}$$

where $k = 0, 1, 2, 3, \ldots, (n-1)$.

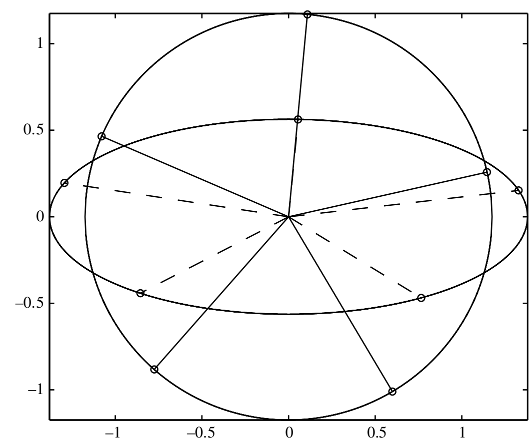
Proof. An application of the generalized De Moivre formula for powers yields

$$\begin{aligned} &\left[r_p^{\frac{1}{n}}\left(\operatorname{cosp}\left(\frac{\theta_p + 2k\pi/\sqrt{|p|}}{n}\right) + i\ \operatorname{sinp}\left(\frac{\theta_p + 2k\pi/\sqrt{|p|}}{n}\right)\right)\right]^n \\ &\quad = r_p\big(\operatorname{cosp}(\theta_p + 2k\pi/\sqrt{|p|}) + i\ \operatorname{sinp}(\theta_p + 2k\pi/\sqrt{|p|})\big) \\ &\quad = r_p(\operatorname{cosp}\theta_p + i\ \operatorname{sinp}\theta_p). \end{aligned}$$ ■

FIGURE 8(a) displays the three elliptical ($p = -1/4$) cube roots of $2 + 5i$. The roots determine three sectors of equal area in the root-ellipse: $x^2 + y^2/4 = \|2 + 5i\|_{-1/4}^{2/3} \approx 2.172$. In general, each set of n complex roots on a root-ellipse in $\mathbb{C}_p$ ($p < 0$) partitions the root-ellipse into n sectors of equal area. A comparison of elliptical ($p = -6$) and



(a)



(b)

Figure 8 Illustrations of the elliptical De Moivre theorem

circular ($p = -1$) roots is shown in FIGURE 8(b), which shows five elliptical fifth roots of $1 + 2i$ and the five circular fifth roots of $1 + 2i$.

The lack of periodicity in the trigonometric functions when $p > 0$ and when $p = 0$ permits a slightly modified De Moivre theorem for the computation of nth roots of parabolic and hyperbolic complex numbers.

THEOREM 3. (ROOTS OF PARABOLIC AND HYPERBOLIC COMPLEX NUMBERS) *For $z \in \mathbb{C}_p$ ($p > 0$ or $p = 0$) and n a positive integer,*

$$z^{\frac{1}{n}} = \left[r_p(\operatorname{cosp}\theta_p + i\ \operatorname{sinp}\theta_p)\right]^{\frac{1}{n}} = r_p^{\frac{1}{n}}\left(\operatorname{cosp}\left(\frac{\theta_p}{n}\right) + i\ \operatorname{sinp}\left(\frac{\theta_p}{n}\right)\right).$$

In the next four examples we illustrate the disparate outcomes that result from the lack of periodicity in the parabolic ($p = 0$) and hyperbolic ($p > 0$) trigonometric functions. FIGURE 9(a) displays the two square roots of $2 + 3i$ in $\mathbb{C}_0$. The two vertical lines drawn are actually the parabolic circle whose radius is given by $\|2 + 3i\|_0^{1/2}$. In FIGURE 9(b), we find only a single cube root of $3 + 2i$ in $\mathbb{C}_1$. There are no others. This cube root lies in branch I on the root-hyperbola given by $|x^2 - y^2| = \|3 + 2i\|_1^{2/3} \approx 1.71$. Since the square of any hyperbolic or parabolic complex number lands in branch I, then the cube of the number winds up back in its original branch. In light of this observation, we see that each hyperbolic and each parabolic complex number has exactly one cube root. Moreover, when n is an odd positive integer, every hyperbolic and parabolic complex number has exactly one nth root.

In FIGURE 9(c), we illustrate the existence of four fourth roots of $4 - 3i$ when $p = 1$. And finally, in FIGURE 9(d), we display the four distinct square roots of $2 + 2\sqrt{3}\,i$ with $p = 1/4$. Note that $\|2 + 2\sqrt{3}\,i\|_{1/4} = 1$ implies that all of the roots lie on the unit hyperbola in $\mathbb{C}_{1/4}$. When n is an even positive integer then hyperbolic complex numbers in branch I have exactly four nth roots (one in each branch), and hyperbolic complex numbers in the other branches have no nth roots. Similarly, every parabolic complex number in branch I has two nth roots when n is even, and parabolic complex numbers in branch II have no even nth roots. The total number of nth roots of a generalized complex number is summarized in TABLE 1. The situation becomes more complicated when looking for solutions of polynomials that are defined over parabolic or hyperbolic complex number systems [**1**].

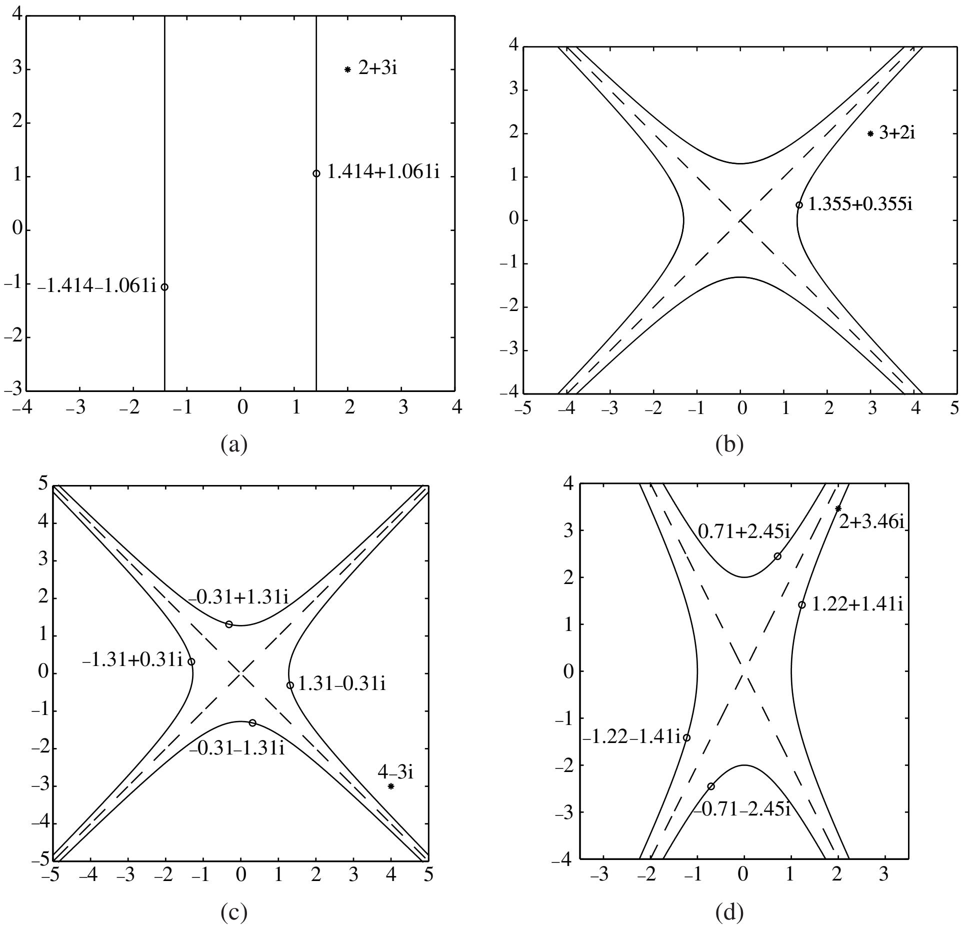


Figure 9 Illustrations of the generalized De Moivre theorem for parabolic and hyperbolic roots

TABLE 1: Number of nth Roots of $z \in \mathbb{C}_p$

$p < 0$		n roots
$p = 0$	$z \in$ branch I	$z \in$ branch II
n even	2 nth roots	0 nth roots
n odd	1 nth root	1 nth root
$p > 0$	$z \in$ branch I	$z \in$ branch II, III, or IV
n even	4 nth roots	0 nth roots
n odd	1 nth root	1 nth root

Functions of a generalized complex variable

At this point, one might wonder about a generalization of the theory of complex analytic functions. We make a few brief observations about analyticity in $\mathbb{C}_p$.

The p-derivative of a function f of a generalized complex variable $z \in \mathbb{C}_p$ is defined, as usual, by

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z},$$

provided this limit exists independent of the manner in which $\Delta z \to 0$, excluding approaches on which the quotient is not defined. Recall that a function $f = u + iv$ of an ordinary complex variable $z = x + iy$ is analytic on a region, D, if and only if it satisfies

$$i\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} \quad \text{on } D.$$

Suppose that f is a function of a generalized complex variable, then we say that $f = u + iv$ is p-analytic when its real and imaginary parts satisfy generalized Cauchy-Riemann equations,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = p\frac{\partial v}{\partial x}.$$

Moreover, if these partial derivatives are continuous, then the real and imaginary parts of f are order-p harmonic:

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{p}\frac{\partial^2 u}{\partial y^2} = 0, \qquad \frac{\partial^2 v}{\partial x^2} - \frac{1}{p}\frac{\partial^2 v}{\partial y^2} = 0.$$

In the special case $p = -1$, the real and imaginary parts of f satisfy Laplace's equation, and when $p = 1$, the real and imaginary parts of f satisfy a wave equation. Study referred to analytic functions of a dual variable ($p = 0$) as *synectic* functions.

As an example, consider the exponential e^z in $\mathbb{C}_p$:

$$e^z = e^{x+iy} = e^x e^{iy} = e^x(\operatorname{cosp} y + i\ \operatorname{sinp} y) = u + iv.$$

Since the real and imaginary parts are

$$u = e^x \operatorname{cosp} y \qquad v = e^x \operatorname{sinp} y$$

and since the derivatives of cosp and sinp are given by

$$\frac{d}{dy}(\operatorname{cosp} y) = p\ \operatorname{sinp} y \qquad \frac{d}{dy}(\operatorname{sinp} y) = \operatorname{cosp} y,$$

it can be verified that the Cauchy-Riemann equations hold. Thus e^z is p-analytic.

Integration is defined on rectifiable curves. When $f(z)$ is differentiable and C is a closed curve, it can be shown that

$$\oint_C f(z)\,dz = 0$$

for all spaces $\mathbb{C}_p$. However, Cauchy's integral formula does not hold in the parabolic or hyperbolic complex planes, as discussed by Deakin [**3**].

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Permutation Notations

Permutations can be thought of as shuffles or rearrangements, but they are most easily described as one-to-one functions from a set onto itself. For example, take your two hands and match them as follows: pinkies to pinkies, fourth fingers to thumbs, index fingers to middle fingers. Numbering the fingers one through five the same way on each hand (and cheating a little by calling the thumb a finger), we get a function:

$$f(1) = 4, \quad f(2) = 3, \quad f(3) = 2, \quad f(4) = 1, \quad f(5) = 5.$$

This is all one needs for certain applications. But authors Deutsch, Johnson, and Thanatipanonda use *line notation*, which is simply a list of the values of the function in order: 43215. This works well for permutations of small sets, but author Scully uses line notation where some elements have names like k and $k - 1$. For clarity, brackets and parentheses can be used: [4, 3, 2, 1, 5].

A longer version of line notation uses two lines in a before-and-after display, like this:

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 3 & 2 & 1 & 5 \end{bmatrix}.$$

Both line notation and function notation obscure some valuable information about the cycles that occur upon repeated applications of a permutation. This is apparent in *cycle* notation. Our finger permutation would be written as $(14)(23)(5)$ or more simply $(14)(23)$. This notation is read as "1 goes to 4, which goes back to 1; 2 goes to 3, which goes back to 2; 5 goes to 5." When an element is omitted, it is understood to stay fixed.

Both the two-line notation and cycle notation were introduced by Cauchy in 1815. You can read a translated excerpt from his paper in *The History of Mathematics: A Reader*, edited by John Fauvel and Jeremy Gray, Macmillan Press in association with The Open University, 1987, pp. 506–507.

Incidentally, the finger permutation described above is the starting point for "compound eensy-weensy spider."