Bounding Intersections of Orbit Sets with Curves

Joel D. Dreibelbis
1 Introduction
   • Orbit Sets and Curves
   • Strategy for Obtaining an Upper Bound on |Orb(\vec{q}) \cap C|
   • General Results on Orbit Sets Intersecting Varieties

2 Orbit Sets Induced by a Linear Map with Eigenvalues in \( \mathbb{R} \)
   • Preliminaries
   • Real Eigenvalues Theorem

3 Orbit Sets Induced by a Linear Map with Eigenvalues in \( \mathbb{Q}_p \)
   • Overview of \( \mathbb{Q}_p \)
   • Algorithm by Example

4 Conclusion
   • Summary
   • Open Questions
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The Orbit Set

- $\Phi : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is a linear map defined as 
  $\Phi(x, y) := (ax + by, cx + dy)$ for some $a, b, c, d \in \mathbb{R}$.
- $\vec{q} \in \mathbb{R}^2$ is the initial point in the orbit set.
- $\text{Orb}_\Phi(\vec{q}) := \{\Phi^n(\vec{q}) \mid n \in \mathbb{N}\} \subset \mathbb{R}^2$ is the orbit set of $\vec{q}$ under $\Phi$.

Note: $\Phi^n := \Phi(\Phi^{n-1})$. 
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Example of an Orbit Set

Let $\Phi(x, y) := (2x, -3y)$, $\vec{q} := (6, 1)$, and $P_n := \Phi^n(\vec{q})$.

Then, $\text{Orb}_\Phi(\vec{q}) = \{(6, 1), (12, -3), (24, 9), (48, -27), (96, 81), \ldots\}.$
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The Eigenvalues of $\Phi$

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- $M := \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \text{Mat}_{2 \times 2}(\mathbb{R})$ is the associated matrix of $\Phi$ since
  $\Phi^n(x, y) = M^n \cdot \begin{bmatrix} x \\ y \end{bmatrix}$ where $\Phi^n := \Phi^{n-1} \circ \Phi$.

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J. Dreibelbis  
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J. Dreibelbis  Bounding Intersections of Orbit Sets with Curves
Review of Curves

- $C := \left\{(x, y) \in \mathbb{R}^2 \right\mid \sum_{\substack{i+j \leq d \\ i,j \geq 0}} a_{i,j} x^i y^j = 0 \right\}$ where $a_{i,j} \in \mathbb{R}$ and $a_{i,d-i} \neq 0$ for some $i$ is a curve of degree $d$. For example, $Z(6xy + x^2y + 5y + 7)$ is a curve of degree 3.

- A curve $C$ of degree $d$ has at most $(d + 1)(d + 2)/2$ coefficients with $\delta := (d + 1)(d + 2)/2 - 1 = d(d + 3)/2$ "degrees of freedom." For example, a conic (curve of degree 2) has five degrees of freedom (five "sufficiently independent" points determine a unique conic).
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Famous Theorem of Curves Intersecting

**Theorem (Bézout 1779)**

If $C_1$ is a curve of degree $d_1$, $C_2$ is a curve of degree $d_2$, and $|C_1 \cap C_2| < \infty$, then $|C_1 \cap C_2| \leq d_1 d_2$. 
The Main Questions

Question (Non-trivial)

Can an orbit set intersect a curve in more points than the "degrees of freedom" for the curve (but in only finitely many points)?

Question (Uniformity)

Is there a uniform upper bound, dependent only on the degree of a curve, for $|\text{Orb}_\Phi(\vec{q}) \cap C|$ over all linear maps and all starting points when the intersection is finite?
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Let $\Phi(x, y) := (2x, -3y)$, $\bar{q} := (6, 1)$, and $P_n := \Phi^n(\bar{q})$.

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Observations

- Lines have 2 degrees of freedom.
- There is an orbit set and a line with 3 points of intersection.
- For a curve $C$ of degree $d$ that has finite intersection with an orbit set induced by a linear map $\Phi$, there is a uniform bound that depends on the eigenvalues of $\Phi$:

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where $\delta = d(d + 3)/2$ and $N = \delta + 1$. 

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Examples with Infinite Intersection

- Let $\Phi(x, y) := (2x, 2y)$, $\vec{q} := (1, 2)$, and $C := \mathbb{Z}(y - 2x)$. Then $\Phi^n(\vec{q}) \in C$ for all $n$ and $|\text{Orb}_\Phi(\vec{q}) \cap C| = \infty$.

- Let $\theta := \sqrt{2}\pi$,
  $\Phi(x, y) := (x \cos(\theta) - y \sin(\theta), x \sin(\theta) + y \cos(\theta))$, $\vec{q} := (1, 0)$, and $C := \mathbb{Z}(x^2 + y^2 - 1)$. Then $\Phi^n(\vec{q}) \in C$ for all $n$ and $|\text{Orb}_\Phi(\vec{q}) \cap C| = \infty$.

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- If $|\text{Orb}_\Phi(\vec{q}) \cap C| = \infty$ then there is some $k \in \mathbb{Z}^+$ and a curve $D \subseteq C$ so that $\Phi^k(D) = D$. 

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Strategy

- Parameterize the coordinates of the points, $P_n$, in the orbit set, $\text{Orb}_\Phi(\vec{q})$, as $P_n = (f_1(n), f_2(n))$ for suitable functions $f_i(n)$.

  - If $P_n \in C = \mathbb{Z} \left( \sum_{i+j \leq d \atop i,j \geq 0} a_{i,j} x^i y^j \right)$ then
    $$\sum_{i+j \leq d \atop i,j \geq 0} a_{i,j} (f_1(n))^i (f_2(n))^j = 0.$$  

  - The last summation will be a polynomial-exponential sum in $n$.
  
  - The poly-exp sum, when the eigenvalues of $\Phi$ are real, generates either 1 or 2 real differentiable functions that detect all of the integer zeroes.
  
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A polynomial-exponential sum is a summation with the form

$$E(x) := \sum_{i=1}^{m} (P_i(x)b_i^x)$$

where $P_i(x) \in k[x]$ and $b_i \in k$ for some field $k$.

- The order of a poly-exp sum is $m + \sum_{i=1}^{m} \deg(P_i)$.
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Recurrence Sequences

- A linear recurrence sequence of order $N$ over a field $k$ is a sequence, $\{a_n\}_{n \in \mathbb{N}}$, of the form

$$a_{n+N} := \alpha_1 a_{n+N-1} + \alpha_2 a_{n+N-2} + \cdots + \alpha_N a_n$$

for $n \geq 0$ with initial values

$$(a_0, a_1, \ldots, a_{N-1}) := (\beta_0, \beta_1, \ldots, \beta_{N-1})$$

for some $\alpha_i, \beta_i \in k$ and $\alpha_N \neq 0$. (or just $N$-ary recurrence sequence over $k$ for short).

- Characteristic polynomial $x^N - \alpha_1 x^{N-1} - \alpha_2 x^{N-2} - \cdots - \alpha_N$ with roots $r_1, r_2, \ldots, r_m$ with $r_i$ having multiplicity $m_i$ so that

$$a_n = \sum_{i=1}^{m} \left( \sum_{j=1}^{m_i} c_{i,j} n^{j-1} \right) r_i^n.$$

- A recurrence sequence is non-degenerate if it takes on the value 0 finitely many times.
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- Characteristic polynomial $x^N - \alpha_1 x^{N-1} - \alpha_2 x^{N-2} - \cdots - \alpha_N$ with roots $r_1, r_2, \ldots, r_m$ with $r_i$ having multiplicity $m_i$ so that

$$a_n = \sum_{i=1}^{m} \left( \sum_{j=1}^{m_i} c_{i,j} n^{j-1} \right) r_i^n.$$

- A recurrence sequence is non-degenerate if it takes on the value 0 finitely many times.
### Recurrence Sequences

- A linear recurrence sequence of order $N$ over a field $k$ is a sequence, $\{a_n\}_{n \in \mathbb{N}}$, of the form
  \[ a_{n+N} := \alpha_1 a_{n+N-1} + \alpha_2 a_{n+N-2} + \cdots + \alpha_N a_n \]
  for $n \geq 0$ with initial values
  \( (a_0, a_1, \ldots, a_{N-1}) := (\beta_0, \beta_1, \ldots, \beta_{N-1}) \) for some $\alpha_i, \beta_i \in k$ and $\alpha_N \neq 0$. (or just $N$-ary recurrence sequence over $k$ for short).

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Example of a Recurrence Sequence of Order 2

- \( a_{n+2} := a_{n+1} + a_n \) with \( (a_0, a_1) := (0, 1) \).
- \( \{a_n\}_{n \in \mathbb{N}} = \{0, 1, 1, 2, 3, 5, 8, \ldots \} \).
- Characteristic polynomial is \( x^2 - x - 1 \) whose roots are \( \frac{1 \pm \sqrt{5}}{2} \).

\[
a_n = c_1 \left( \frac{1 + \sqrt{5}}{2} \right)^n + c_2 \left( \frac{1 - \sqrt{5}}{2} \right)^n
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a_0 = 0
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Connections to Linear Recurrences

- Given the orbit set problem \((\Phi, \vec{q}, C)\), there is a linear recurrence \(\{a_n\}_{n \in \mathbb{N}}\) so that \(a_n = 0 \iff \Phi^n(\vec{q}) \in C\).

- If \(C\) has degree \(d\) then the linear recurrence will have order at most \(\delta + 1\).

- Uniform bounds already exist for the number of zeroes in a linear recurrences of order \(N\) (Schlickewei, ranging from triply exponential in \(N\) to the most recent, doubly exponential result, about 20 years).
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Skolem-Mahler-Lech Theorem

Theorem (Skolem-Mahler-Lech 1933-1935-1953)

If \( \{a_n\}_{n \in \mathbb{N}} \) is a recurrence sequence of complex numbers, then the set of all integers \( n \) such that \( a_n = 0 \) is the union of a finite number of arithmetic sequences.
An arithmetic sequence of natural numbers is a sequence, \( \{a_n\}_{n \in \mathbb{N}} \), of the form
\[
a_n := s + nt
\]
for some fixed \( s, t \in \mathbb{N} \) and with \( n \in \mathbb{N} \).

- If \( t = 0 \), then the arithmetic sequence is a singleton.
- If \( t \neq 0 \), then the arithmetic sequence is said to be a full arithmetic sequence (contains infinitely many numbers).
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Ternary Recurrence Theorems

Theorem (Beukers 1991)

If \( \{a_n\}_{n \in \mathbb{N}} \) is a non-degenerate ternary recurrence sequence of rational numbers, then there are at most 6 integers \( n \) such that \( a_n = 0 \).

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If \( \{a_n\}_{n \in \mathbb{N}} \) is a non-degenerate ternary recurrence sequence of complex numbers, then there are at most 61 integers \( n \) such that \( a_n = 0 \).
Introduction
Orbit Sets Induced by a Linear Map with Eigenvalues in \( \mathbb{R} \)
Orbit Sets Induced by a Linear Map with Eigenvalues in \( \mathbb{Q}_p \)
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Theorem (Schlickewei 2000)

If \( \{a_n\}_{n \in \mathbb{N}} \) is a non-degenerate \( N \)-ary recurrence sequence of rational numbers, then there are at most \( (2N)^{35N^3} \) integers \( n \) such that \( a_n = 0 \).

Theorem (D. 2010)

For \( N > 1 \), if \( \{a_n\}_{n \in \mathbb{N}} \) is a non-degenerate \( N \)-ary recurrence sequence of real numbers whose characteristic roots are all real, then there are at most \( 2N - 3 \) integers \( n \) such that \( a_n = 0 \).
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Orbit Sets and Varieties

Analyze \( \{ n \in \mathbb{N} \mid \Phi^n(\vec{q}) \in W \} \) where \( \Phi : V \to V \), \( \vec{q} \in V \), and \( W \) is a subvariety of \( V := \bigcap_{i=1}^{k} Z(P_i(\vec{x})) \).

**Theorem (Bell 2006)**

Let \( V \) be an affine variety over a field \( k \) of characteristic 0. Let \( \vec{q} \) be a point in \( V \) and \( \Phi \) an automorphism of \( V \). If \( W \) is a subvariety of \( V \) then the set \( \{ n \in \mathbb{N} \mid \Phi^n(\vec{q}) \in W \} \) is a finite union of arithmetic sequences.
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Theorem (Bell, Ghioca, Tucker 2009)

Let $\Phi : V \rightarrow V$ be an étale endomorphism of any quasiprojective variety defined over $\mathbb{C}$. Then for any subvariety $W$ of $V$, and for any point $\vec{q} \in V$ the set $\{n \in \mathbb{N} \mid \Phi^n(\vec{q}) \in W\}$ is a finite union of arithmetic sequences.

Theorem (D. 2010)

If the eigenvalues of a linear map $\Phi : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ are real, $\vec{q} \in \mathbb{R}^2$, $C \subset \mathbb{R}^2$ is a curve of degree $d$, and $|\text{Orb}_\Phi(\vec{q}) \cap C|$ is finite, then $|\text{Orb}_\Phi(\vec{q}) \cap C| \leq d^2 + 3d - 1$. 
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By change of basis, we may assume that $\Phi(x, y) = (\lambda_1 x, \lambda_2 y)$ or $\Phi(x, y) = (\lambda x + y, \lambda y)$ (using the Jordan form for the matrix associated to $\Phi$).

It will be shown that if $|\text{Orb}_\Phi(\vec{q}) \cap C|$ is finite then $|\text{Orb}_\Phi(\vec{q}) \cap C| \leq d^2 + 3d - 1$. 

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Main Lemma

Lemma (D. 2010)

If \( E(x) := \sum_{i=1}^{m} (P_i(x)b_i^x) \) is a poly-exp sum over \( \mathbb{R} \) with \( \text{ord}(E) := N \geq 2 \) then the number of integer zeroes of \( E(x) \) is at most \( 2N - 3 \).

When all \( b_i \) are positive, then the number of integer zeroes of \( E(x) \) is at most \( N - 1 \).

When there are at least two \( b_i \) of opposite sign, then the number of integer zeroes of \( E(x) \) is at most \( 2(N - 1) \). By an inductive argument, we may lower the bound by 1, so that there are at most \( 2(N - 1) - 1 = 2N - 3 \) integer zeroes.
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Lemma (D. 2010)

There at most \(2 \deg(P(x)) + 1\) real solutions to either (1) or (2), where \(a, b, c, d \in \mathbb{R}^+\) with \(P(x)\) and \(Q(x)\) monic polynomials of the same degree.

\[
P(x)a^x + cb^x = 0 \quad (1)
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This lemma is a generalization of the fact that there is at most one solution to either \(2^x + 3^x = 0\) or \(2^x - 3^x = 0\).
Savings of One Lemma

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Rolle’s Theorem

Theorem (Rolle 1691)

For a differentiable function $f : \mathbb{R} \to \mathbb{R}$ and any non-negative integer $r$, $\#Z(f(x)) \leq r + \#Z(f^{(r)}(x))$.

For a real, differentiable poly-exp sum, Rolle’s Theorem provides an upper bound on the number of real zeroes.
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Main Lemma: Example

Let $E(x) := (x^2)3^x + (-2)^x + (-5x)7^x$. Then $E(x)$ has order $N = 6$ ($3 + 2 + 0 + 1$, number of exponential terms plus the polynomial degree sum). Since there is one positive exponential base value and one negative, we parameterize $E(x)$ as

$$E_1(x) := (2x)^23^{2x} + (-2)^{2x} - 5(2x)7^{2x} = 4x^29^x + 4^x - 10x49^x$$

$$E_2(x) := (2x + 1)^23^{2x+1} + (-2)^{2x+1} - 5(2x + 1)7^{2x+1}$$

$$= 3(4x^2 + 4x + 1)9^x - (2)4^x - 7(10x + 5)49^x.$$  

*Note: $E_1(n) = E(2n)$ and $E_2(n) = E(2n + 1)$ for all $n \in \mathbb{Z}$.***
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To find an upper bound on the number of zeroes of $E_i(x)$, we repeat the steps of

1. Dividing by an exponential term ($+0$).
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Main Lemma: Example

By definition, we have

$$\# Z(E_1(x)) = \# Z(4x^2 9^x + 4^x - 10x49^x)$$
$$\# Z(E_2(x)) = \# Z(3(4x^2 + 4x + 1)9^x - 24^x - 7(10x + 5)49^x) .$$

Dividing by an exponential term does not change the number of zeroes,

$$\# Z(E_1(x)) = \# Z(4x^2 (9/49)^x + (4/49)^x - 10x)$$
$$\# Z(E_2(x)) = \# Z(3(4x^2 + 4x + 1) (9/49)^x - 2 (4/49)^x - 7(10x + 5)) .$$
Main Lemma: Example

By definition, we have

\[
\#Z(E_1(x)) = \#Z \left( 4x^2 9^x + 4^x - 10x49^x \right)
\]
\[
\#Z(E_2(x)) = \#Z \left( 3(4x^2 + 4x + 1)9^x - 24^x - 7(10x + 5)49^x \right).
\]

Dividing by an exponential term does not change the number of zeroes,

\[
\#Z(E_1(x)) = \#Z \left( 4x^2 \left( \frac{9}{49} \right)^x + \left( \frac{4}{49} \right)^x - 10x \right)
\]
\[
\#Z(E_2(x)) = \#Z \left( 3(4x^2 + 4x + 1) \left( \frac{9}{49} \right)^x - 2 \left( \frac{4}{49} \right)^x - 7(10x + 5) \right).
\]
Main Lemma: Example

After taking two derivatives and applying Rolle’s Theorem, we have

\[
\#Z (E_1(x)) \leq 2 + \#Z (Q(x) (9/49)^x + c_2 (4/49)^x) \\
\#Z (E_2(x)) \leq 2 + \#Z (R(x) (9/49)^x - 2c_2 (4/49)^x).
\]

Rewriting these sums so that the Savings of One Lemma can be applied,

\[
\#Z (E_1(x)) \leq 2 + \#Z \left( \frac{1}{c_1} Q(x) (9/49)^x + \frac{c_2}{c_1} (4/49)^x \right) \\
\#Z (E_2(x)) \leq 2 + \#Z \left( \frac{1}{3c_1} R(x) (9/49)^x - \frac{2c_2}{3c_1} (4/49)^x \right).
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After taking two derivatives and applying Rolle’s Theorem, we have

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\]
Main Lemma: Example

- Applying the Savings of One Lemma where the degree of the monic polynomials is 2, the number of real zeroes to either the first or second poly-exp sum is $2(2) + 1 = 5$.
- Finally, $\#Z(E_1(x)) + \#Z(E_2(x)) \leq (2 + 2) + 5 = 9$.
- So there are at most 9 real zeroes to either $E_1(x)$ or $E_2(x)$. Therefore, $E(x)$, a poly-exp sum of order 6, has at most 9 integer zeroes with $9 = 2(6) - 3$ (the number expected based the main lemma).
Main Lemma: Example

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- Applying the Savings of One Lemma where the degree of the monic polynomials is 2, the number of real zeroes to either the first or second poly-exp sum is $2(2) + 1 = 5$.
- Finally, $\#Z(E_1(x)) + \#Z(E_2(x)) \leq (2 + 2) + 5 = 9$.
- So there are at most 9 real zeroes to either $E_1(x)$ or $E_2(x)$. Therefore, $E(x)$, a poly-exp sum of order 6, has at most 9 integer zeroes with $9 = 2(6) - 3$ (the number expected based the main lemma).
D. Real Eigenvalues Theorem

Theorem (D. 2010)

If the eigenvalues of a linear map $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ are real, $\vec{q} \in \mathbb{R}^2$, $C \subset \mathbb{R}^2$ is a curve of degree $d$, and $|\text{Orb}_\Phi(\vec{q}) \cap C|$ is finite, then $|\text{Orb}_\Phi(\vec{q}) \cap C| \leq d^2 + 3d - 1$. 
Suppose $\Phi(x, y) := (\lambda_1 x, \lambda_2 y)$.

Let $C$ be a curve of degree $d$ defined over $\mathbb{R}$ with

$$C := \mathbb{Z} \left( \sum_{i+j \leq d, i,j \geq 0} a_{i,j} x^i y^j \right)$$

for some $a_{i,j} \in \mathbb{R}$ and $\vec{q} := (q_1, q_2) \in \mathbb{R}^2$.

Then, $\Phi^n(\vec{q}) = (q_1 \lambda_1^n, q_2 \lambda_2^n)$.

The points in the orbit set $\text{Orb}_\Phi(\vec{q})$ are $P_n := (q_1 \lambda_1^n, q_2 \lambda_2^n)$ for $n \geq 0$. 
Suppose $\Phi(x, y) := (\lambda_1 x, \lambda_2 y)$.
Let $C$ be a curve of degree $d$ defined over $\mathbb{R}$ with
\[
C := \sum_{i,j \geq 0, i+j \leq d} a_{i,j} x^i y^j
\]
for some $a_{i,j} \in \mathbb{R}$ and $\vec{q} := (q_1, q_2) \in \mathbb{R}^2$.
Then, $\Phi^n(\vec{q}) = (q_1 \lambda_1^n, q_2 \lambda_2^n)$.
The points in the orbit set $\text{Orb}_{\Phi}(\vec{q})$ are $P_n := (q_1 \lambda_1^n, q_2 \lambda_2^n)$ for $n \geq 0$. 
Suppose \( \Phi(x, y) := (\lambda_1 x, \lambda_2 y) \).

Let \( C \) be a curve of degree \( d \) defined over \( \mathbb{R} \) with

\[
C := \mathbb{Z} \left( \sum_{\substack{0 \leq i, j \leq d}} a_{i,j} x^i y^j \right)
\]
for some \( a_{i,j} \in \mathbb{R} \) and \( \vec{q} := (q_1, q_2) \in \mathbb{R}^2 \).

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The points in the orbit set \( \text{Orb}_\Phi(\vec{q}) \) are \( P_n := (q_1 \lambda_1^n, q_2 \lambda_2^n) \) for \( n \geq 0 \).
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Then, $\Phi^n(\vec{q}) = (q_1 \lambda_1^n, q_2 \lambda_2^n)$.

The points in the orbit set $\text{Orb}_\Phi(\vec{q})$ are $P_n := (q_1 \lambda_1^n, q_2 \lambda_2^n)$ for $n \geq 0$. 
If $P_n \in C$ then

\[ 0 = \sum_{i+j \leq d \atop i,j \geq 0} a_{i,j} (q_1 \lambda_1^n)^i (q_2 \lambda_2^n)^j = \sum_{i+j \leq d \atop i,j \geq 0} a_{i,j} q_1^i q_2^j (\lambda_1^n \lambda_2^n)^n = \sum_{i+j \leq d \atop i,j \geq 0} b_{i,j} c_{i,j}^n := E(n) \]

where $b_{i,j} := a_{i,j} q_1^i q_2^j$ and $c_{i,j} := \lambda_1^i \lambda_2^j$.

Since $\text{ord}(E) \leq \frac{(d+1)(d+2)}{2}$ by counting the number of distinct exponential terms and each polynomial coefficient is degree zero, we have at most $2\text{ord}(E) - 3 = d^2 + 3d - 1$ integer solutions to the poly-exp equation. Therefore, $|\text{Orb}_\Phi(\overline{q}) \cap C| \leq d^2 + 3d - 1$. 
If $P_n \in C$ then

$$0 = \sum_{i+j \leq d \atop i,j \geq 0} a_{i,j} (q_1^n \lambda_1^i)(q_2^n \lambda_2^j) = \sum_{i+j \leq d \atop i,j \geq 0} a_{i,j} q_1^i q_2^j (\lambda_1^i \lambda_2^j)^n = \sum_{i+j \leq d \atop i,j \geq 0} b_{i,j} c_{i,j}^n := E(n)$$

where $b_{i,j} := a_{i,j} q_1^i q_2^j$ and $c_{i,j} := \lambda_1^i \lambda_2^j$.

Since $\text{ord}(E) \leq \frac{(d+1)(d+2)}{2}$ by counting the number of distinct exponential terms and each polynomial coefficient is degree zero, we have at most $2\text{ord}(E) - 3 = d^2 + 3d - 1$ integer solutions to the poly-exp equation. Therefore, $|\text{Orb}_\Phi(\vec{q}) \cap C| \leq d^2 + 3d - 1$. 
D. Real Theorem: Proof Part 1

If \( P_n \in C \) then

\[
0 = \sum_{i+j \leq d \atop i,j \geq 0} a_{i,j} (q_1 \lambda_1^n)^i (q_2 \lambda_2^n)^j = \sum_{i+j \leq d \atop i,j \geq 0} a_{i,j} q_1^i q_2^j (\lambda_1^i \lambda_2^j)^n = \sum_{i+j \leq d \atop i,j \geq 0} b_{i,j} c_{i,j}^n := E(n)
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where \( b_{i,j} := a_{i,j} q_1^i q_2^j \) and \( c_{i,j} := \lambda_1^i \lambda_2^j \).

Since \( \text{ord}(E) \leq \frac{(d+1)(d+2)}{2} \) by counting the number of distinct exponential terms and each polynomial coefficient is degree zero, we have at most \( 2\text{ord}(E) - 3 = d^2 + 3d - 1 \) integer solutions to the poly-exp equation. Therefore, \( |\text{Orb}_\Phi(\vec{\eta}) \cap C| \leq d^2 + 3d - 1 \).
D. Real Theorem: Proof Part 2

Suppose \( \Phi(x, y) := (\lambda x + y, \lambda y) \).

Let \( C \) be a curve of degree \( d \) defined over \( \mathbb{R} \) with

\[
C := Z \left( \sum_{\substack{i+j \leq d \\ i, j \geq 0}} a_{i,j} x^i y^j \right)
\]

for some \( a_{i,j} \in \mathbb{R} \) and \( \vec{q} := (q_1, q_2) \in \mathbb{R}^2 \).

Then, \( \Phi^n(\vec{q}) = (q_1 \lambda^n + nq_2 \lambda^{n-1}, q_2 \lambda^n) \).

The points in the orbit set \( \text{Orb}_\Phi(\vec{q}) \) are \( P_n := (q_1 \lambda^n + nq_2 \lambda^{n-1}, q_2 \lambda^n) \) for \( n \geq 0 \).
Suppose $\Phi(x, y) := (\lambda x + y, \lambda y)$. Let $C$ be a curve of degree $d$ defined over $\mathbb{R}$ with

$$C := Z \left( \sum_{i+j \leq d \atop i,j \geq 0} a_{i,j} x^i y^j \right)$$

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J. Dreibeibis  Bounding Intersections of Orbit Sets with Curves
Suppose $\Phi(x, y) := (\lambda x + y, \lambda y)$. Let $C$ be a curve of degree $d$ defined over $\mathbb{R}$ with

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D. Real Theorem: Proof Part 2

If \( P_n \in C \) then

\[
0 = \sum_{i+j \leq d, \ i,j \geq 0} a_{i,j}(q_1 \lambda^n + nq_2 \lambda^{n-1})^i(q_2 \lambda^n)^j = \sum_{i+j \leq d, \ i,j \geq 0} a_{i,j}(q_1 \lambda^n + \frac{q_2}{\lambda} n\lambda^n)^i(q_2 \lambda^n)^j
\]

\[
= \sum_{i+j \leq d, \ i,j \geq 0} a_{i,j}q_2^i(q_1 + \frac{q_2}{\lambda} n)^i \lambda^{ni} \lambda^{nj} = \sum_{i+j \leq d, \ i,j \geq 0} a_{i,j}q_2^i(q_1 + \frac{q_2}{\lambda} n)^i (\lambda^{i+j})^n
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\[
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\]

where \( P_k \) is a polynomial in \( n \) of at most degree \( k \).
If $P_n \in C$ then

\[
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D. Real Theorem: Proof Part 2

\( E(n) \) is a poly-exp sum of order at most

\[
N \leq d + 1 + \sum_{k=0}^{d} k = d + 1 + \frac{d(d + 1)}{2} = \frac{d^2 + 3d + 2}{2}.
\]

Therefore, there are at most \( 2N - 3 = d^2 + 3d - 1 \) integer solutions to \( E(n) = 0 \). Consequently, \( \left| \text{Orb}_\Phi(\vec{q}) \cap C \right| \leq d^2 + 3d - 1. \]
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§
The $p$-adic ordinal

- For an integer $s$ and a prime number $p$, we define $\text{ord}_p(s) := k$ where $k$ is the largest power of $p$ dividing $s$. For example,

  \[
  \text{ord}_p(p^2) = 2
  \]

  \[
  \text{ord}_5(4 \cdot 5^3 + 5^5 + 3 \cdot 5^6) = \text{ord}_5(5^3 \cdot 404) = 3
  \]

  \[
  \text{ord}_7(19) = 0.
  \]

- Since an element $q \in \mathbb{Q}$ may be written as $q = s_1/s_2$ for some $s_i \in \mathbb{Z}$, we may extend the function $\text{ord}_p(\cdot)$ to $\mathbb{Q}$ by

  \[
  \text{ord}_p(q) := \text{ord}_p(s_1) - \text{ord}_p(s_2).
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The $p$-adic ordinal

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$$\text{ord}_p(q) := \text{ord}_p(s_1) - \text{ord}_p(s_2).$$
Another useful function is the $p$-adic absolute value $|\cdot|_p : \mathbb{Q} \to \mathbb{Q}$ where $|q|_p := p^{-\text{ord}_p(q)}$.

$\mathbb{Q}_p$ is the completion of $\mathbb{Q}$ under $|\cdot|_p$ (whereas $\mathbb{R}$ is the completion of $\mathbb{Q}$ under $|\cdot|$).

Properties of these functions for $q_1, q_2 \in \mathbb{Q}_p$:

(a) $\text{ord}_p(q_1 + q_2) \geq \min(\text{ord}_p(q_1), \text{ord}_p(q_2))$;

(b) $|q_1 + q_2|_p \leq \max(|q_1|_p, |q_2|_p)$ (much stronger than the triangle inequality);

(c) $s \in \mathbb{Z}_p \iff \text{ord}_p(s) \geq 0$. 

The $p$-adic absolute value

Overview of $\mathbb{Q}_p$ Algorithm by Example
The $p$-adic absolute value

- Another useful function is the $p$-adic absolute value $| \cdot |_p : \mathbb{Q} \to \mathbb{Q}$ where $|q|_p := p^{-\text{ord}_p(q)}$.
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J. Dreibelbis

Bounding Intersections of Orbit Sets with Curves
The $p$-adic absolute value

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  (c) $s \in \mathbb{Z}_p \iff \text{ord}_p(s) \geq 0$. 
Fix a prime number $p$. An element $q \in \mathbb{Q}_p$ has the unique form, for some $L \in \mathbb{Z}$,

$$q := \sum_{i=L}^{\infty} q_i p^i$$

where each $q_i \in \mathbb{Z}$ satisfies $0 \leq q_i < p$.

Here are some notable special cases (with $M$ an integer):

- $\sum_{i=L}^{M} q_i p^i \in \mathbb{Q}$
- $\sum_{i=0}^{\infty} q_i p^i \in \mathbb{Z}_p$
- $\sum_{i=0}^{M} q_i p^i \in \mathbb{Z}$. 
Structure of $\mathbb{Q}_p$

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- Here are some notable special cases (with $M$ an integer):

$$\sum_{i=L}^{M} q_i p^i \in \mathbb{Q}$$

$$\sum_{i=0}^{\infty} q_i p^i \in \mathbb{Z}_p$$

$$\sum_{i=0}^{M} q_i p^i \in \mathbb{Z}.$$
The $p$-adic ordinal on $\mathbb{Q}_p$

- The $p$-adic ordinal and absolute value can be extended to $\mathbb{Q}_p$.
- For example,

$$\text{ord}_p(p^{-3} + p^{-2} + p^{-1} + 1 + p + \cdots) = -3$$

$$\text{ord}_7(1/49) = \text{ord}_7(1) - \text{ord}_7(49) = -2$$

$$\text{ord}_5(1 + 5 + 5^2 + 5^3 + \cdots) = \text{ord}_5(1/(1 - 5)) = 0.$$
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Strategy

- Parameterize the coordinates of the points, $P_n$, in the orbit set, $\text{Orb}_\Phi(\vec{q})$, as $P_n = (f_1(n), f_2(n))$ for suitable functions $f_i(n)$.

- If $P_n \in C = \mathbb{Z} \left( \sum_{i+j \leq d \atop i,j \geq 0} a_{i,j} x^i y^j \right)$ then $\sum_{i+j \leq d \atop i,j \geq 0} a_{i,j} (f_1(n))^i (f_2(n))^j = 0$.

- The last summation, when the eigenvalues of $\Phi$ are in $\mathbb{Q}_p$, generates at most $p^2$ $p$-adic analytic functions.

- The number of integer zeroes for each $p$-adic analytic function is then bounded by Strassman’s Theorem using the $\text{ord}_p(\cdot)$ values of the coefficients.
Strategy

- Parameterize the coordinates of the points, \( P_n \), in the orbit set, \( \text{Orb}_\Phi(\vec{q}) \), as \( P_n = (f_1(n), f_2(n)) \) for suitable functions \( f_i(n) \).

- If \( P_n \in C = \mathbb{Z} \left( \sum_{i+j \leq d, \ i,j \geq 0} a_{i,j}x^iy^j \right) \) then \( \sum_{i+j \leq d, \ i,j \geq 0} a_{i,j}(f_1(n))^i(f_2(n))^j = 0. \)

- The last summation, when the eigenvalues of \( \Phi \) are in \( \mathbb{Q}_p \), generates at most \( p^2 \) \( p \)-adic analytic functions.

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The $p^2$ $p$-adic analytic functions

Suppose $\Phi : \mathbb{Q}_p^2 \rightarrow \mathbb{Q}_p^2$ has associated matrix $M$ which is invertible modulo $p\mathbb{Z}_p$. We may also assume that $M$ is in Jordan form.

Using the $p$-adic logarithm and exponential functions when $M \equiv I \pmod{p\mathbb{Z}_p}$ we find that $\Phi^n(\vec{q}) = M^n\vec{q} = \exp_p(\log_p(M^n))\vec{q} = (f_1(n), f_2(n))$.

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The $p^2$ $p$-adic analytic functions

- If $M \not\equiv I \pmod{p\mathbb{Z}_p}$ then $M^k \equiv I \pmod{p\mathbb{Z}_p}$ for some $k \leq p^2$.
- For example, $M := \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$.
- Over $\mathbb{Q}_5$, $M^4 \equiv I \pmod{5\mathbb{Z}_5}$.
- Then we can partition the orbit set into $k$ subsets:
  $$\text{Orb}_\Phi(q) = \bigcup_{i=0}^{k-1} \text{Orb}_{\Phi^k}(\Phi^i(q)).$$
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Strassman’s Theorem

Theorem (Strassman 1928)

Let \( F(n) = \sum_{k=0}^{\infty} a_k n^k \in \mathbb{Q}_p[[n]] \) with \( \text{ord}_p(a_k) \to \infty \) so that \( F(n) \) converges for all \( n \in \mathbb{Z}_p \). Let \( N \) be defined by:

(i) \( \text{ord}_p(a_N) = \min \{ \text{ord}_p(a_k) \mid k \in \mathbb{N} \} \);

(ii) \( \text{ord}_p(a_N) < \text{ord}_p(a_k) \) for all \( k > N \);

then \( F : \mathbb{Z}_p \to \mathbb{Q}_p \) has at most \( N \) zeroes in \( \mathbb{Z}_p \) (and thus at most \( N \) zeroes in \( \mathbb{Z} \)).

\( N \) is the index of the last coefficient with the minimum \( \text{ord}(\cdot) \) value.
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$N$ is the index of the last coefficient with the minimum $\text{ord}(\cdot)$ value.
Suppose $F(n) = \sum_{k=0}^{\infty} a_k n^k \in \mathbb{Q}_p[[n]]$. Plot the points $(k, \text{ord}_p(a_k))$ and form the lower convex hull. Identify $N$. 

![Newton Polygon Graph](image.png)
Newton Polygon

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![Newton Polygon Diagram]

$\text{ord}_p(a_k)$

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Summary of the Algorithm for \( \mathbb{Q}_p \)

- Once \( \Phi, \tilde{q}, \) and \( d \) are chosen, then we only have freedom to choose coefficients for our curve.

- To obtain the maximum number of integer zeroes in our analytic function, we must choose the coefficients to minimize the \( \text{ord}_p(\cdot) \) values of the initial coefficients.

- This leads to a system of linear equations. If the linear equations are linearly independent (which is the expectation if the intersection is finite) then there are at most \( \delta \) zeroes.

- If the original power series generates \( p^2 \) analytic functions, then there is expected to be \( p^2 \delta \) integer zeroes (so then \( |\text{Orb}_\Phi(\tilde{q}) \cap C| \leq p^2 \delta \)).
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Let \( p := 5, \Phi(x, y) := (6x + 5y, 6y) \), \( \vec{q} := (1, 2) \), and \( C := \mathbb{Z}(\gamma + \alpha x + \beta y) \) defined over \( \mathbb{Q}_p \) and, without loss of generality, we may assume that \( \min(\text{ord}_p(\alpha), \text{ord}_p(\beta)) = 0 \).

Then, \( \text{Orb}_\Phi(\vec{q}) = \{(1, 2), (16, 12), (156, 72), (1296, 432), \ldots \} \).

It will be shown that \( |\text{Orb}_\Phi(\vec{q}) \cap C| \leq \delta = 2 \) for all curves \( C \) of degree \( d = 1 \) over \( \mathbb{Q}_p \).
Example

Let $p := 5$, $\Phi(x, y) := (6x + 5y, 6y)$, $\vec{q} := (1, 2)$, and $C := \mathbb{Z}(\gamma + \alpha x + \beta y)$ defined over $\mathbb{Q}_p$ and, without loss of generality, we may assume that $\min(\text{ord}_p(\alpha), \text{ord}_p(\beta)) = 0$. Then, $\text{Orb}_\Phi(\vec{q}) = \{(1, 2), (16, 12), (156, 72), (1296, 432), \ldots \}$. It will be shown that $|\text{Orb}_\Phi(\vec{q}) \cap C| \leq \delta = 2$ for all curves $C$ of degree $d = 1$ over $\mathbb{Q}_p$. 
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We find that there are parameterizing functions $f_i(n)$ so that $\Phi^n(1, 2) = (f_1(n), f_2(n))$.

Write $f_1(n) := \sum_{k=0}^{\infty} b_k n^k$ and $f_2(n) := \sum_{k=0}^{\infty} c_k n^k$.

Analyzing the parameterizing functions, it can be shown that

$$\text{ord}_p(b_k) \geq k \cdot \frac{p-2}{p-1}$$

and

$$\text{ord}_p(c_k) \geq k \cdot \frac{p-2}{p-1}.$$

For $k \geq 3$ and $p = 5$, we have $\min(\text{ord}_p(b_k), \text{ord}_p(c_k)) \geq 3$.

For $k \geq 3$ and $p = 5$, if $\alpha, \beta \in \mathbb{Z}_p$ then

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Example

- The first four coefficients of $f_1(n)$:
  
  $b_0 = 1$
  $b_1 = 3p + 0p^2 + 1p^3 + 1p^4 + b_{1,5}$
  $b_2 = 2p^3 + 0p^4 + b_{2,5}$
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Example

If $P_n \in C$ then

$$0 = \gamma + \alpha f_1(n) + \beta f_2(n)$$

$$= \gamma + b_0 + c_0 + \sum_{k=1}^{\infty} (\alpha b_k + \beta c_k)n^k$$

$$:= \sum_{k=0}^{\infty} d_k n^k$$

$$:= F(n)$$

Choose $\alpha$ and $\beta$ so that $\text{ord}_p(d_1) = \text{ord}_p(d_2)$. Then it is also possible to choose the constant term $\gamma$ so that $\text{ord}_p(d_0) = \text{ord}_p(d_1)$. With these choices, we see that $N \geq 2$.

Next, we will show that $\text{ord}_p(d_k) > \text{ord}_p(d_2)$ for $k > 2$ which implies that $N = 2$. 
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Example

A closer look at the initial coefficients of $F(n)$:

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d_1 = \alpha b_1 + \beta c_1 \\
= \alpha(3p + 0p^2 + 1p^3 + 1p^4 + b_{1,5}) + \beta(2p + 4p^2 + 3p^3 + 0p^4 + c_{1,5}) \\
\equiv (3\alpha + 2\beta)p + (4\beta)p^2 \pmod{p^3}
\]

\[
d_2 = \alpha b_2 + \beta c_2 \\
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Introduction
Orbit Sets Induced by a Linear Map with Eigenvalues in $\mathbb{R}$
Orbit Sets Induced by a Linear Map with Eigenvalues in $\mathbb{Q}_p$
Conclusion

Example

So that $\text{ord}_p(d_1) = 2 = \text{ord}_p(d_2)$ we need:

$$3\alpha + 2\beta \equiv 0 \pmod{p}$$

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Main Results

Theorem (D. 2010)

For $N > 1$, if $\{a_n\}_{n \in \mathbb{N}}$ is a non-degenerate $N$-ary recurrence sequence of real numbers whose characteristic roots are all real, then there are at most $2N - 3$ integers $n$ such that $a_n = 0$.

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If the eigenvalues of a linear map $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ are real, $\vec{q} \in \mathbb{R}^2$, $C \subset \mathbb{R}^2$ is a curve of degree $d$, and $|\text{Orb}_\Phi(\vec{q}) \cap C|$ is finite, then $|\text{Orb}_\Phi(\vec{q}) \cap C| \leq d^2 + 3d - 1$.

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J. Dreibelbis  
Bounding Intersections of Orbit Sets with Curves
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J. Dreibelbis  Bounding Intersections of Orbit Sets with Curves
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Find a sharp, upper bound $B$ where $\delta + 1 \leq B \leq 2\delta - 1$ for $|\text{Orb}_\Phi(\vec{q}) \cap C|$ when $\Phi$ has real eigenvalues.

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If $\Phi : \mathbb{R}^g \rightarrow \mathbb{R}^g$ is a linear map with real eigenvalues, $\vec{q} \in \mathbb{R}^g$,

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with $a_{i_1,\ldots,i_g} \in \mathbb{R}$, and $|\text{Orb}_\Phi(\vec{q}) \cap H|$ is finite then there are at most $2\delta - 1$ points in the intersection.
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If \( \Phi : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) is a linear map with eigenvalues in \( \mathbb{Q}_p \), \( \vec{q} \in \mathbb{Q}_p^2 \), \( C \) is a curve of degree \( d \) over \( \mathbb{Q}_p \), and \( |\text{Orb}_\Phi(\vec{q}) \cap C| \) is finite then there are at most \( p^2 \cdot \delta \) points in the intersection.

For a map \( \Phi : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) with \( \Phi(x, y) := (P_1(x, y), P_2(x, y)) \) find a uniform upper bound for \( |\text{Orb}_\Phi(\vec{q}) \cap C| \) depending only on the degree of \( C \) and the degrees of \( P_1 \) and \( P_2 \).
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