§4.1 – Extreme Values of Functions

Definition: Let $D$ be the domain of a function $f$.

(i) $f$ has an **absolute maximum** at $x = c$ if $f(c) \geq f(x)$ for all $x \in D$. $f(c)$ is called the **maximum value** of $f$ on $D$.

(ii) $f$ has an **absolute minimum** at $x = c$ if $f(c) \leq f(x)$ for all $x \in D$. $f(c)$ is called the **minimum value** of $f$ on $D$.

The maximum and minimum values of $f$ are called the **extreme values** of $f$.

**Example 1:** Find the $x$-coordinates of any absolute min/max of the function given in the graph below.

![Graph showing a function with several points marked]

**Example 2:** Find the absolute min/max of $f(x) = x^2$.

**Example 3:** Find the absolute min/max of $f(x) = x^2$ defined on the closed interval $[-1, 2]$.

**The Extreme Value Theorem:** If $f$ is continuous on a closed interval $[a, b]$, then there exists numbers $c, d \in [a, b]$ such that $f(c)$ is an absolute max value and $f(d)$ is an absolute min value.
**Definition:** Let $f$ be a function.

(i) $f$ has a **local maximum** at $x = c$ if $f(c) \geq f(x)$ for all $x$ ‘near’ $c$.

(ii) $f$ has a **local minimum** at $x = c$ if $f(c) \leq f(x)$ for all $x$ ‘near’ $c$.

‘Near’ means that there exists an open interval $I$ containing $c$ such that the inequality is true for all values $x \in I$.

**Fermat’s Theorem:** If $f$ has a local min/max at $c$ and if $f'(c)$ exists, then $f'(c) = 0$.

**Definition:** A **critical number** of $f$ is any number $c$ in the domain of $f$ such that either $f'(c) = 0$ or $f'(c)$ does not exist.

**Example 4:** Find all critical numbers of $f(x) = x^{1/3}(x+4)$.

**The Closed Interval Method:** Given a continuous function $f$ on a closed interval $[a,b]$, the absolute minimum/maximum can be determined by following the steps below.

*Step 1.* Find all critical numbers of $f$ in $(a,b)$.

*Step 2.* Calculate the functional value(s) of the number(s) in Step 1.

*Step 3.* Calculate $f(a)$ and $f(b)$.

*Step 4.* The largest and smallest values in steps 2 and 3 is, respectively, the absolute minimum and absolute maximum.

**Example 5:** Give the $x$ and $y$ coordinates of the absolute min/max(s) of $f(x) = x^4 - 6x^2 + 3$ on $[-3,2]$.

**Example 6:** Give the $x$ and $y$ coordinates of the absolute min/max(s) of $f(x) = e^{x^2-x}$ on $[-1,1]$. 
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![Graph of a function](image)

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