§4.3 – Monotonic Functions and the First Derivative Test

Recall from Section 1.1
Definition: Let \( f \) be a function and \( I \) an interval contained in the domain of \( f \).
   
   (i) \( f \) is increasing on \( I \) if \( f(x_1) < f(x_2) \) whenever \( x_1 < x_2 \) in \( I \).

   (ii) \( f \) is decreasing on \( I \) if \( f(x_1) > f(x_2) \) whenever \( x_1 < x_2 \) in \( I \).

Recall from Section 4.2
The Mean Value Theorem (MVT): If \( f \) is continuous on \([a,b]\) and differentiable on \((a,b)\), then there exists a number \( c \) in \((a,b)\) such that

\[
 f'(c) = \frac{f(b) - f(a)}{b - a}.
\]

Increasing/Decreasing Test (IDT): Let \( f \) be a function defined on \([a,b]\).

   (i) If \( f'(x) > 0 \) for all \( x \in (a,b) \), then \( f \) is increasing on \([a,b]\).

   (ii) If \( f'(x) < 0 \) for all \( x \in (a,b) \), then \( f \) is decreasing on \([a,b]\).

Example 1: Let \( f(x) = 4x^4 - 8x^2 \). Find the intervals on which \( f \) is …

   (a) … increasing.

   (b) … decreasing.
The First Derivative Test (FDT): Let \( c \) be a critical number of a continuous function \( f \).

(i) If \( f' \) changes from positive to negative at \( c \), then \( f \) has a local max at \( c \).

(ii) If \( f' \) changes from negative to positive at \( c \), then \( f \) has a local min at \( c \).

(iii) If \( f' \) does not change signs at \( c \) (that is, \( f' \) is positive on both sides of \( c \) or negative on both sides of \( c \)), then \( f \) has no local extremum at \( c \).

Example 2: Given \( f(x) = x^{1/3}(x + 4) \).

(a) Find intervals of increase or decrease.

(b) Find the \( x \) and \( y \) coordinates of all local mins/maxs, if any.

Example 3: Given \( f(x) = \frac{1}{1 + e^{-x}} \).

(a) Find all vertical and horizontal asymptotes.

(b) Find the intervals on which \( f \) is increasing/decreasing.

(c) Find the \( x \) and \( y \) coordinates of all local mins/maxs, if any.
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Recall from Section 1.1

Definition: Let \( f \) be a function and \( I \) an interval contained in the domain of \( f \).

(i) \( f \) is increasing on \( I \) if \( f(x_1) < f(x_2) \) whenever \( x_1 < x_2 \) in \( I \).

(ii) \( f \) is decreasing on \( I \) if \( f(x_1) > f(x_2) \) whenever \( x_1 < x_2 \) in \( I \).

A function that is either increasing or decreasing on \( I \) is called monotonic on \( I \).

Recall from Section 4.2

The Mean Value Theorem (MVT): If \( f \) is continuous on \([a,b]\) and differentiable on \((a,b)\), then there exists a number \( c \) in \((a,b)\) such that

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Example 2: Given $f(x) = x^{1/3} (x + 4)$.

(a) Find where $f$ is increasing/decreasing.

(b) Find the $x$ and $y$ coordinates of all local mins/maxs, if any.
Example 3: Given $f(x) = \frac{1}{1+e^{-x}}$.

(a) Find all vertical and horizontal asymptotes.
(b) Find the intervals on which $f$ is increasing/decreasing.
(c) Find the $x$ and $y$ coordinates of all local mins/maxs, if any.