Topics in Logic, Set Theory and Computability

Homework Set #1

Due Wednesday 3/21 at 3pm (by email or in person at 08-3234)

Exercises from Handouts
1-B
2-B
4-G
4-K
4-M
4-N
5-I
5-L
6-B
6-E
7-A-2
Solutions to Homework #1

1-B. Let \( X \subseteq Y \), \( Y \subseteq Z \), and \( Z \subseteq X \). Prove that \( X = Y = Z \).

*Proof:*
Need to show two things: (1) \( X = Y \) and (2) \( Y = Z \).

(1) Show \( X = Y \). Since we are given that \( X \subseteq Y \), then we need only show that \( Y \subseteq X \). Let \( b \in Y \). [Now show \( b \in X \).] Since \( b \in Y \) and \( Y \subseteq Z \), then \( b \in Z \). Since \( b \in Z \) and \( Z \subseteq X \), then \( b \in X \).

(2) Show \( Y = Z \). Since we are given that \( Y \subseteq Z \), then we need only show that \( Z \subseteq Y \). Let \( c \in Z \). [Now show \( c \in Y \).] Since \( c \in Z \) and \( Z \subseteq X \), then \( c \in X \). Since \( c \in X \) and \( X \subseteq Y \), then \( c \in Y \).

\[\blacksquare\]

Out of: 5 | Class Average: 98 % | Class Median: 100 %

2-B. An adjective is *autological* if it has the property it denotes and *heterological* if it does not have the property that it denotes. Which of the following words are autological and which are heterological?

(a) “English” (b) “Anglais” (c) “long”
(d) “monosyllabic” (e) “polysyllabic” (f) “hyphenated”
(g) “bevowelled” (h) “autological” (i) “heterological”

*Answer*
(a) The word “English” is autological.
(b) The word “Anglais” is French for “English”, so it is heterological.
(c) The word “long” is short, so it is heterological.
(d) The word “monosyllabic” is polysyllabic, so it is heterological.
(e) The word “polysyllabic” is polysyllabic, so it is autological.
(f) The word “hyphenated” is not hyphenated, so it is heterological.
(g) The word “bevowelled” contains vowels, so it is autological.
(h) The word “autological” can be taken, without contradiction, to be either autological or heterological. To see why, consider the following.
   - If we say that “autological” is autological, and then ask if it applies to itself, then yes, it does, and thus is autological.
   - If we say that “autological” is heterological, and then ask if it applies to itself, then no, it does not, and thus is heterological.
(i) The word “heterological” is neither. To see why, consider the following.
   - If we say that “heterological” is heterological, then it must be autological (leading to a contradiction).
   - If we say that “heterological” is autological, then the answer is no, it is heterological (again leading to a contradiction, because if it describes itself, it is autological).

[This not-as-well-known paradox is attributed to Grelling-Nelson (first proposed in 1908).]

Out of: 4 | Class Average: 96 % | Class Median: 95 %

4-G. Prove that if \( A \cup B = A \cup C \) and \( A \cap B = A \cap C = \emptyset \), then \( B = C \).

*Proof:*
\(\Rightarrow\): Suppose \( x \in B \). [Show \( x \in C \).]
   Then \( x \in A \cup B \).
   Since \( A \cap B = \emptyset \) and \( x \in B \), then \( x \notin A \).
   Since \( A \cup B = A \cup C \) and \( x \in A \cup B \), then \( x \in A \cup C \).
Since $A \cap C = \emptyset$ and $x \notin A$, then $x \in C$.
Thus $B \subseteq C$.

$\iff$ A similar argument will show $C \subseteq B$.
Thus $B = C$.

4-K. Let $X = \{ \{1,2\}, \{1\}, \{1,0\} \}$. Determine each of the following.

(a) $U \cap X$  (b) $\cap X$  (c) $\cup U \cup X$  (d) $\cap \cap X$  (e) $\cup U \cup X$  (f) $\cap \cap X$

**Answer**
Let $A = \{ \{1,2\}, \{1\} \}$ and $B = \{ \{1,0\} \}$, then $X = \{ A, B \}$.

(a) $U \cap X = A \cup B = \{ \{1,2\}, \{1\}, \{1,0\} \}$
(b) $\cap X = A \cap B = \emptyset$
(c) $\cup U \cup X = \cup \{ \{1,2\}, \{1\}, \{1,0\} \} = \{ 0, 1, 2 \} = 3$
(d) $\cap \cap X = \cap \emptyset$ is not defined
(e) $\cup U \cup X = \cap \{ \{1,2\}, \{1\}, \{1,0\} \} = \{ 1 \}$
(f) $\cap \cap X = \cap \emptyset = \emptyset$

4-M. Express the set 5 using only the four symbols ‘{’, ‘}’, ‘∅’ and ‘,’. Your answer should contain a minimum number of the symbols to represent the set 5. How many symbols did you use?

**Answer**
$0 = \emptyset$
$1 = 0 \cup \{0\}$
$= \emptyset \cup \{\emptyset\}$
$= \{\emptyset\}$
$2 = 1 \cup \{1\}$
$= \{\emptyset\} \cup \{\{\emptyset\}\}$
$= \{\emptyset, \{\emptyset\}\}$
$3 = 2 \cup \{2\}$
$= \{\emptyset, \{\emptyset\}\} \cup \{\{\emptyset, \{\emptyset\}\}\}$
$= \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$
$4 = 3 \cup \{3\}$
$= \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\} \cup \{\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}\}$
$= \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}\}$
$5 = 4 \cup \{4\}$
$= \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}\} \cup \{\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}\}\}$
$= \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}, \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}\}\}$

Let $a_n$ be the number of symbols (the four given in the problem) needed to write out the set $n$. Using the list above, we see that $a_0 = 1$, $a_1 = 3$, $a_2 = 7$, $a_3 = 15$, $a_4 = 31$, and $a_5 = 63$.

In general, we have the recurrence relation $a_{n+1} = 2a_n + 1$ with initial condition $a_0 = 1$. The solution is $a_n = 2^n - 1$. 

Out of: 6 Class Average: 86 % Class Median: 85 %
4-N. (Distributive Laws) Let $X$ be a set and $C$ a collection of sets. Prove each of the following.

(a) $X \cap U C = U \{ X \cap A : A \in C \}$

(b) If $C$ is nonempty, then $X \cup \cap C = \cap \{ X \cup A : A \in C \}$.

**Answer**

(a) $X \cap U C = U \{ X \cap A : A \in C \}$

**Proof:**

Let $D$ be the collection $D = \{ X \cap A : A \in C \}$. 

$\Rightarrow$: Suppose $z \in X \cap UC$. [Show $z \in UC$.]

Then $z \in X$ and $z \in UC$.

Since $z \in UC$, there exists $B \in C$ such that $z \in B$.

Since $z \in X$ and $z \in B$, then $z \in X \cap B$.

Since $z \in X \cap B$ and $X \cap B \in D$, then $z \in U D$. 

$\Leftarrow$: Suppose $z \in U D$. [Show $z \in X \cap UC$.]

Then there exists $B \in C$ such that $z \in X \cap B$.

Then $z \in X$ and $z \in B$.

Since $z \in B$ and $B \in C$, then $z \in UC$.

Since $z \in X$ and $z \in UC$, then $z \in X \cap UC$.

Thus $X \cap UC = U D$. 

(b) If $C$ is nonempty, then $X \cup \cap C = \cap \{ X \cup A : A \in C \}$.

**Proof:**

Let $D$ be the collection $D = \{ X \cup A : A \in C \}$. Since $C$ is nonempty, then $D$ must also be nonempty.

$\Rightarrow$: Suppose $z \in X \cup \cap C$. [Show $z \in \cap D$.]

Then $z \in X$ or $z \in \cap C$.

**Case 1:** Suppose $z \in X$.

By Exercise 4-F(a), $X \subseteq X \cup A$ and so $z \in X \cup A$ for any $A \in C$.

In particular, $z \in X \cup A$ for every $A \in C$.

Thus $z \in \cap D$.

**Case 2:** Suppose $z \in \cap C$.

Then $z \in A$ for every $A \in C$.

By Exercise 4-F(a) and Theorem 4-8(b), $A \subseteq A \cup X = X \cup A$ and so $z \in X \cup A$ for every $A \in C$.

Thus $z \in \cap D$.

In either case we have $z \in \cap D$.

$\Leftarrow$: Suppose $z \in \cap D$. [Show $z \in X \cup \cap C$.]

Then $z \in X \cup A$ for every $A \in C$.

If $z \in X$, then we use Exercise 4-F(a) to note that $X \subseteq X \cup C$. This implies that $z \in X \cup \cap C$.

If $z \in \cap C$, then since $\cap C \subseteq X \cup \cap C$ and so $z \in X \cup \cap C$.

In either case we have $z \in X \cup \cap C$.

Thus $X \cup \cap C = \cap D$. 

| Problem 4-N(a) | Out of: 5 | Class Average: 95% | Class Median: 95% |
| Problem 4-N(b) | Out of: 5 | Class Average: 87% | Class Median: 95% |
5-I. Construct 3 – \((\cup \mathcal{P}1)\).

**Answer**

Step 1:

\[ \mathcal{P}1 = \mathcal{P}\{0\} = \{ \emptyset, \{0\} \} = \{ 0, 1 \} = 2 \]

Step 2:

\[ \cup \mathcal{P}1 = \cup \{ 0, 1 \} = 0 \cup 1 = 1 \]

Step 3:

\[ 3 - (\cup \mathcal{P}1) = 3 - 1 = \{0,1,2\} - \{0\} = \{1,2\} \]

5-L. Let \(X\) and \(Y\) be sets. Prove \(X \subseteq Y\) iff \(\mathcal{P}X \subseteq \mathcal{P}Y\).

Recall that \(A \subseteq X \iff A \in \mathcal{P}X\).

**Proof:**

We first show: If \(X \subseteq Y\) then \(\mathcal{P}X \subseteq \mathcal{P}Y\). Suppose \(X \subseteq Y\) and \(A \in \mathcal{P}X\). By definition of the power set, \(A \in \mathcal{P}X\) implies \(A \subseteq X\). Using our hypothesis, \(X \subseteq Y\), then \(A \subseteq Y\). Again, by definition of the power set, we have \(A \in \mathcal{P}Y\).

For the converse, there are two proofs that can be used. The first is an “element proof” similar to the previous approach and the second is a direct proof without appeal to elements inside \(X\).

**Claim:** If \(\mathcal{P}X \subseteq \mathcal{P}Y\) then \(X \subseteq Y\).

**Proof #1:** Let \(\mathcal{P}X \subseteq \mathcal{P}Y\). Suppose \(x \in X\). Then \(\{x\} \subseteq X\). By definition of the power set, we have \(\{x\} \in \mathcal{P}X\). Using our hypothesis \(\mathcal{P}X \subseteq \mathcal{P}Y\), then \(\{x\} \in \mathcal{P}Y\). Again, by definition of the power set, we have \(\{x\} \subseteq Y\). This implies that \(x \in Y\). Thus, \(X \subseteq Y\). ■

**Proof #2:** Let \(\mathcal{P}X \subseteq \mathcal{P}Y\). Since \(X \subseteq X\), then \(X \in \mathcal{P}X\). Since \(X \in \mathcal{P}X\) and \(\mathcal{P}X \subseteq \mathcal{P}Y\) (by hypothesis), then \(X \in \mathcal{P}Y\). By the definition of the power set, then \(X \subseteq Y\). ■

6-B. Show that each of the following sets cannot serve as the definition for the ordered pair \((x, y)\).

(a) \((x, y) = \{x, y\}\)

(b) \((x, y) = \{x, \{y\}\}\)

**Answer**

(a) Since \((0,1) = \{0,1\} = \{1,0\} = (1,0)\), then this is not a suitable definition.

(b) Since \((\{0\},1) = \{\{0\}, \{1\}\} = \{\{1\}, \{0\}\} = ((1),0)\) and \(0 \neq 1\), then this is not a suitable definition.
6-E. Prove that the Cartesian product is not commutative.

**Answer**
That is, find sets $X$ and $Y$ such that $X \times Y \neq Y \times X$.

Try $X = 1 = \{ 0 \}$ and $Y = 2 = \{ 0, 1 \}$.

Then $X \times Y = \{ (0,0), (0,1) \}$ and $Y \times X = \{ (0,0), (1,0) \}$.

Thus $X \times Y \neq Y \times X$.

7-A-2. List all the relations on (a) the set 1; (b) the set 2.

**Answer**
(a) Since $1 \times 1 = \{ (0,0) \}$, then the two relations on 1 are $\emptyset$ and $1 \times 1$.

(b) Since $2 \times 2 = \{ (0,0), (0,1), (1,0), (1,1) \}$, then the sixteen relations on 2 are given in the following table.

<table>
<thead>
<tr>
<th>Number of elements</th>
<th>Relations</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$\sigma_1 = \emptyset$</td>
</tr>
</tbody>
</table>
| 1                 | $\sigma_2 = \{ (0,0) \}$  
|                   | $\sigma_3 = \{ (0,1) \}$  
|                   | $\sigma_4 = \{ (1,0) \}$  
|                   | $\sigma_5 = \{ (1,1) \}$  |
| 2                 | $\sigma_6 = \{ (0,0), (0,1) \}$  
|                   | $\sigma_7 = \{ (0,0), (1,0) \}$  
|                   | $\sigma_8 = t_2$  
|                   | $\sigma_9 = \{ (0,1), (1,0) \}$  
|                   | $\sigma_{10} = \{ (0,1), (1,1) \}$  
|                   | $\sigma_{11} = \{ (1,0), (1,1) \}$  |
| 3                 | $\sigma_{12} = 2 \times 2 - \{ (1,1) \}$  
|                   | $\sigma_{13} = t_2 \cup \{ (0,1) \}$  
|                   | $\sigma_{14} = t_2 \cup \{ (1,0) \}$  
|                   | $\sigma_{15} = 2 \times 2 - \{ (0,0) \}$  
| 4                 | $\sigma_{16} = 2 \times 2$  |