Let $X$ be a continuous random variable with probability density function

$$f(x) = \begin{cases} \frac{3}{16}(4-x^2) & 0 < x < 2 \\ 0 & \text{elsewhere} \end{cases}$$

Find

A. $P(0<X<1) = \int_0^1 \frac{3}{16}(4-x^2)\,dx = 0.6875$

B. The mean

$$EX = \int_0^2 \frac{3}{16}x(4-x^2)\,dx = 0.75$$

C. The variance and the standard deviation.

$$EX^2 = \int_0^2 \frac{3}{16}x^2(4-x^2)\,dx = 0.8$$

$$\sigma^2 = 0.8 - 0.75^2 = 0.2375$$

$$\sigma = 0.4873$$

D. The cumulative distribution function.

$$F(x) = \int_0^x \frac{3}{16}(4-t^2)\,dt = \left[\frac{3}{16}(4t - \frac{1}{3}t^3)\right]_0^x = \frac{3}{16}(4x - \frac{1}{3}x^3)$$

$$F(X) = \begin{cases} 0 & x < 0 \\ \frac{3}{16}(4x - \frac{1}{3}x^3) & 0 < x < 2 \\ 1 & x > 2 \end{cases}$$
2. The line width for semiconductor manufacturing is assumed to be normally distributed with a mean of 0.5 micrometers and a standard deviation of 0.05 micrometers.

A. What is the probability that a line width is greater than 0.62 micrometer?
(7 points)

\[
P(W > 0.62) = \left( Z > \frac{0.62 - 0.50}{0.05} \right) = P(Z > 2.4) = 1 - P(Z \leq 2.4) = 1 - .9918 = .0082
\]

\[
\text{normalcdf}(0.62, 10^99, 0.5, 0.05) = .0082
\]

B. What is the probability a line width is between 0.47 and 0.63 meter?
(7 points)

\[
P(0.47 \leq W \leq 0.63) = P \left( \frac{0.47 - 0.50}{0.05} \leq Z \leq \frac{0.63 - 0.50}{0.05} \right)
\]
\[
= P(-.6 \leq Z \leq 2.6) = .9953 - .2743 = .721
\]

\[
\text{normalcdf}(0.47, 0.63, 0.50, 0.05) = .721
\]

C. The line width of 90% of the samples is below what value?
(6 points)

\[
P(W \leq c) = 0.9
\]
\[
P\left( Z \leq \frac{c - 0.50}{0.05} \right) = 0.9
\]
\[
\frac{c - 0.50}{0.05} = 1.28
\]
\[
c = 0.50 + 1.28(0.05) = .564
\]
\[
\text{invnorm}(0.9, 0.50, 0.05) = .564
\]

Use the Central Limit Theorem to do this question.

3. The distribution of resistance for a certain type of resistor has mean 100 ohms and standard deviation 6 ohms. Thirty six resistors are connected in series so that the resistances add up. Let \( \bar{R} \) denote the sample mean resistance of the 36 resistors.

A. What is the mean and the standard deviation of \( \bar{R} \) ?(4 points)

\[
\mu_{\bar{R}} = 100, \sigma_{\bar{R}} = 6 / \sqrt{36} = 1
\]
B. Find \( P(\bar{R} < 102) \) (8 points)

\[
P(\bar{R} < 102) = P \left( Z < \frac{102 - 100}{1} \right) = P(Z < 2) = 0.9772
\]

C. What is the probability that the total resistance is at least 3636 ohms? (8 points)

\[
P \left( \sum_{i=1}^{36} R_i \geq 3636 \right) = P(\bar{R} \geq 101) = P \left( Z > \frac{101 - 100}{1} \right)
\]

\[
= P(Z > 1) = 1 - P(Z \leq 1) = 1 - .8413 = .1587
\]

4. The life in hours of a 75 watt light bulb is known to be normally distributed with standard deviation 25 hours. A random sample of 20 bulbs has a mean life \( \bar{x} = 1014 \) hrs.

A. Find a 95% two sided confidence interval for the true mean life of the lightbulb? (8 points)

\[
\left( \bar{x} - z_{\alpha/2} \frac{\sigma}{\sqrt{n}}, \bar{x} + z_{\alpha/2} \frac{\sigma}{\sqrt{n}} \right)
\]

\[
= \left( 1014 - \frac{1.96(25)}{\sqrt{20}}, 1014 + \frac{1.96(25)}{\sqrt{20}} \right)
\]

\[
= (1003.04, 1024.96)
\]

B. Interpret the interval you obtained in part A. (4 points)

The probability that the interval contains the true mean is 0.95.

C. What sample size is needed to obtain an error bound of at most 5 for a 95% confidence interval? (8 points)

\[
n = \left( \frac{z_{\alpha/2} \sigma}{E} \right)^2, n = \left( \frac{1.96(25)}{5} \right)^2 = 96.04, n = 97
\]
5. The life of a battery is known to be normally distributed with a standard deviation of 1.25 hours. A random sample of 10 batteries has a mean life $\bar{x} = 40.8$ hrs. The objective is to perform a test of hypothesis at $\alpha = .05$ to determine whether the mean battery life is significantly greater than 40 hours.

A. State the null and the alternative hypothesis

$H_0 \; \mu = 40$

$H_1 \; \mu > 40$

B. What assumptions are needed for this test?

Normal population with known sd

C. What is the region of rejection?

$Z > 1.645$

D. Compute the value of the test statistic (5 points)

$$Z = \frac{\bar{x} - \mu}{\sigma / \sqrt{n}}$$

$$Z = \frac{40.8 - 40}{1.25 / \sqrt{10}} = 2.02$$

E. Would you reject or fail to reject $H_0$ at $\alpha = .05$? (3 points)

Reject $H_0$

F. Make a statement giving the conclusion about the life of the batteries? What would you tell your boss? Remember he/she probably flunked statistics.

There is statistically significant evidence that the average battery life is more than 40 hours.