1. A continuous random variable $X$ has probability density function

$$f(x) = \begin{cases} \frac{x}{2} & 0 < x < 2 \\ 0 & \text{otherwise} \end{cases}$$

Find

A. $P(0 < X < 1)$ (4 points)

$$P(0 < X < 1) = \int_{0}^{1} \frac{x}{2} \, dx = \left[ \frac{x^2}{4} \right]_{0}^{1} = \frac{1}{4}$$

B. Find the mean of $X$. (4 points)

$$EX = \int_{0}^{2} \frac{x^2}{2} \, dx = \left[ \frac{x^3}{6} \right]_{0}^{2} = \frac{8}{6} = \frac{4}{3}$$

C. Find the standard deviation of $X$. (4 points)

$$EX^2 = \int_{0}^{2} \frac{x^3}{2} \, dx = \left[ \frac{x^4}{8} \right]_{0}^{2} = 2$$

$$\sigma^2 = 2 - \frac{16}{9} = \frac{2}{9}$$

$$\sigma = \sqrt{\frac{2}{9}} = .471$$
D. Find $c$ so that $P(0 < X < c) = \frac{9}{16}$ (4 points)

\[ \int_{0}^{c} \frac{x}{2} \, dx = \frac{c^2}{4} = \frac{9}{16} \]

\[ c^2 = \frac{9}{4} \]

\[ c = \frac{3}{2} \]

2. The finished inside diameter of a piston ring is normally distributed with a mean of 10 centimeters and a standard deviation of .03 cm.

A. What proportion of the rings will have inside diameter exceeding 10.075 cm? (8 points)

\[ P(X > 10.075) = P \left( Z > \frac{10.075 - 10}{.03} \right) = P(Z > 2.5) = 1 - P(Z \leq 2.5) \]

\[ = 1 - .9938 = .0062 \]

B. Below what diameter will 15% of the piston rings fall? (8 points)

\[ P(X < c) = .15 \]

\[ P \left( Z < \frac{c - 10}{.03} \right) = .15 \]

\[ \frac{c - 10}{.03} = -1.036 \]

\[ c = 10 - 1.036(.03) = 9.9689 \]
3. Use the normal approximation to the binomial distribution to do this problem. Make the continuity correction.
A process yields 10% defective items. If 100 items are randomly selected approximate the probability that the number of defectives is at most 8. (12 points)

\[
\mu = 100(.10) = 10 \\
\sigma = \sqrt{100(.10)(.90)} = 3 \\
P(X \leq 8) = P \left( Z < \frac{8.5 - 10}{3} \right) = P(Z < -.5) = .3085
\]

4. Use the Central Limit Theorem to do this problem.

The amount of time that a customer spends waiting at an airport check-in counter is a random variable with mean 8.0 minutes and standard deviation 1.4 minutes. Suppose that a random sample of n = 49 customers is observed. Find the probability that the average time waiting in line for these customers is between 7.8 minutes and 8.1 minutes. (12 points)

\[
\mu_x = 8, \sigma_x = 1.4 / \sqrt{49} = 0.2 \\
P(7.8 < \bar{x} < 8.1) = P \left( \frac{7.8 - 8}{0.2} < Z < \frac{8.1 - 8}{0.2} \right) \\
= P(-1 < Z < .5) = .6915 - .1587 = .5328
\]

5. Wasserman and Wadsworth (1989) discuss a process for the manufacture of steel bolts that continuously feed an assembly line downstream. Historically the thicknesses of these bolts follow a normal distribution with a standard deviation of 1.6 mm. The sample mean for a recent random sample of ten bolts was 9.99.

A. Find a 95% confidence interval for the true mean thickness of the bolts?(7 points)

\[
\left( \bar{x} - \frac{z_{a/2} \sigma}{\sqrt{n}}, \bar{x} + \frac{z_{a/2} \sigma}{\sqrt{n}} \right)
\]

\[
(9.99 - \frac{1.96(1.6)}{\sqrt{10}}, 9.99 + \frac{1.96(1.6)}{\sqrt{10}})
\]

\[
\]

\[
(9.00, 10.98)
\]
B. Does the interval you obtained in A definitely contain the true mean thickness of the bolts? Explain.(3 points)

The probability that the interval contains the true mean thickness of the bolts is 0.95. Thus, there is a 5% chance that the interval will not contain the true mean thickness of the bolts.

6. A manufacturer of sprinkler systems used for fire protection in office buildings claims that the true average system activation temperature is 130 degrees Fahrenheit. A sample of n=9 sprinklers when tested yields a sample activation temperature of 131.08 degrees Fahrenheit. Historically is known that the distribution of activation times is normally distributed with a standard deviation of 1.5 degrees Fahrenheit. The goal is to perform a test of hypothesis at \( \alpha = .05 \) to determine whether the sample data contradicts the manufacturers claim. Answer the following questions.

A. What is the null and the alternative hypothesis?(2 points)

\[ H_0 \ \mu = 130 \]
\[ H_1 \ \mu \neq 130 \]

B. What are the assumptions under which this test of hypothesis is performed?(2 points)

Normal population with a known standard deviations

C. What is the region of rejection?(2 points)

\[
Z = \frac{\bar{x} - \mu_0}{\frac{\sigma}{\sqrt{n}}} \quad Z > 1.96 \quad Z < -1.96
\]

D. Find the value of the test statistic. (4 points)

\[
Z = \frac{131.08 - 130}{1.5 / \sqrt{9}} = 2.16
\]

E. Based on your answers to C and D would you reject or fail to reject \( H_0 \) at \( \alpha = .05 \)? (2 points)

Reject \( H_0 \) at \( \alpha = .05 \). \( Z = 2.16 \) is in the critical region.
F. What conclusion do you come to regarding the manufacturers claim? (2 points)

There is reason to doubt the manufacturers claim.

G. Find the p value for this test.

\[ P(Z < -2.16) + P(Z > 2.16) = .0307 \]

7. The waiting time in hours between successive speeders spotted by a radar trap on a thruway is an exponential random variable with cumulative distribution

\[ F(x) = \begin{cases} 
0 & x < 0 \\
1 - e^{-8x} & x \geq 0 
\end{cases} \]

A. Find the probability of waiting less than 0.2 hr (12 minutes) between successive speeders. (3 points)

\[ F(0.2) = 1 - e^{-8(0.2)} \approx .7981 \]

B. What is the probability density function for this random variable? (6 points)

\[ f(x) = F'(x) = \begin{cases} 
0 & x < 0 \\
8e^{-8x} & x \geq 0 
\end{cases} \]

C. What is the mean of the waiting times between successive speeders? (You need not do any computations) (3 points)

mean waiting time = 1/8 hour
D. What is median of the waiting times between successive speeders?(6 points)

\[ 1 - e^{-8x} = 0.5 \]
\[ e^{-8x} = 0.5 \]
\[ -8x = \ln 0.5 \]
\[ x = .0866 \text{hr} = 5.2 \text{min} \]