1. Let $X$ denote the amount of space occupied by an article placed in a 1-ft³ packing container. The continuous probability density function of $X$ is

$$f(x) = \begin{cases} 
90x^8(1-x) & 0 < x < 1 \\
0 & \text{elsewhere}
\end{cases}$$

Find

A. $P(.2 \leq X \leq .8)$ (6 points)

$$P(.2 \leq X \leq .8) = \int_{.2}^{.8} 90x^8(1-x)\,dx = \int_{.2}^{.8} 90x^8 - 90x^9\,dx$$

$$= 10x^9 - 9x^{10} \bigg|_{.2}^{.8} = .3758$$

B. The mean and the standard deviation of $X$. (12 points)

$$EX = \int_0^1 90x^8(1-x)\,dx = .8182$$

$$EX^2 = \int_0^1 90x^{10}(1-x)\,dx = .6818$$

$$\sigma^2 = .6818 - .8182^2 = .01235$$

$$\sigma = .1111$$

C. The mean and the standard deviation of $Y = 4X - 2$ (6 points)

$$EY = 4(.8182) - 2 = 1.2728$$

$$\sigma_Y = 4(.1111) = .4444$$
D. The cumulative distribution function of $X$. (6 points)

$$
F(x) = \begin{cases} 
0 & x < 0 \\
10x^9 - 9x^{10} & 0 < x < 1 \\
1 & x > 1 
\end{cases}
$$

2. A random variable has cumulative distribution function

$$
F(x) = \begin{cases} 
0 & x < 0 \\
x^2 & 0 \leq x < 4 \\
16 & x \geq 4
\end{cases}
$$

Find
A. $P(X > 3)$ (6 points)

$$
P(X > 3) = 1 - \frac{9}{16} = \frac{7}{16}
$$

B. The median of $X$. (6 points)

$$
x^2 = .5 \\
x^2 = 8 \\
x = \sqrt{8}
$$

C. The probability density function of $X$. (6 points)

$$
f(x) = \begin{cases} 
x & 0 < x < 4 \\
\frac{x}{8} & \text{otherwise}
\end{cases}
$$
3. The tensile strength of paper is modeled by a normal distribution with a mean of 35 pounds per square inch and a standard deviation of 2 pounds per square inch.
A. What is the probability that the strength of a sample is between 34.4 and 36.8 pounds per square inch? (7 points)

\[
P(34.4 < X < 36.8) = P\left(\frac{34.4 - 35}{2} < Z < \frac{36.8 - 35}{2}\right) = P(-0.3 < Z < 0.9) = 0.8159 - 0.3821 = 0.4338
\]

B. Specifications require that the tensile strength exceeds 31 pounds per square inch. What is the probability a sample has to be scrapped? (7 points)

\[
P(X \leq 31) = P\left(Z \leq \frac{31 - 35}{2}\right) = P(Z < -2) = 0.0228
\]

C. What strength is exceeded by the strongest 7% of the samples? (7 points)

\[
P(X > c) = 0.07
P(X \leq c) = 0.93
P\left(Z \leq \frac{c - 35}{2}\right) = 0.93
\]
\[
\frac{c - 35}{2} = 1.48
\]
\[
c = 35 + 1.48(2) = 37.96
\]

D. Suppose it was not known that the tensile strength was normally distributed but it was known that the mean was 35 and the standard deviation was 2. Using Chebychev’s Theorem give a lower bound on the probability that the tensile strength is between 32 and 38 pounds. (6 points)

\[
2k = 3
k = 1.5
\]
\[
\text{bound} = 1 - \frac{1}{1.5^2} = 0.5556
\]
4. Suppose that only 40% of drivers in a certain state wear a seat belt. A random sample of 500 drivers is selected. Use the normal approximation to the binomial distribution with the continuity correction to approximate the probability that at least 220 drivers wear a seat belt. (15 points)

\[
\mu = 500(0.4) = 200 \\
\sigma = \sqrt{500(0.4)(0.6)} = 10.95 \\
P(X \geq 220) = P\left( Z \geq \frac{219.5 - 200}{10.95} \right) = P(Z \geq 1.78) \\
= 1 - P(Z \leq 1.78) = 1 - 0.9625 = 0.0375
\]

5. Given the probability density function

\[
f(x, k, \theta) = \begin{cases} 
\frac{k \theta^k}{x^{k+1}} & x \geq \theta \\
0 & x < \theta 
\end{cases}
\]

Find

\[
P\left(\frac{3\theta}{2} < X < 2\theta\right)
\]

in terms of k. (5 points)

\[
P\left(\frac{1.5}{\theta} < X < \frac{2}{\theta}\right) = \int_{\frac{1.5}{\theta}}^{\frac{2}{\theta}} \frac{k \theta^k}{x^{k+1}} \, dx = -\frac{\theta}{x^k}\bigg|_{\frac{1.5}{\theta}}^{\frac{2}{\theta}} = \frac{1}{2} + \frac{1}{1.5} = \frac{1}{6}
\]

6. The 75th percentile of the distribution that has density function

\[
f(x) = \frac{1}{\theta} e^{-x/\theta}, x > 0
\]
is 2.773. What is the value of \( \theta \)? (5 points)

\[
\int_{0}^{c} \frac{1}{\theta} e^{-x/\theta} \, dx = 0.75 \\
1 - e^{-x/\theta} = 0.75 \\
1 - e^{-2.773/\theta} = 0.75 \\
2.773/\theta = \ln 0.25 \\
\theta = \frac{2.773}{\ln 25} = 2
\]