Fourier Methods for Nonparametric Image Registration

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Abstract
Nonparametric image registration algorithms use deformation fields to define nonrigid transformations relating two images. Typically, these algorithms operate by successively solving linear systems of partial differential equations. These PDE systems arise by linearizing the Euler-Lagrange equations associated with the minimization of a functional defined to contain an image similarity term and a regularizer. Iterative linear system solvers can be used to solve the linear PDE systems, but they can be extremely slow. Some faster techniques based on Fourier methods, multigrid methods, and additive operator splitting, exist for solving the linear PDE systems for specific combinations of regularizers and boundary conditions. In this paper, we show that Fourier methods can be employed to quickly solve the linear PDE systems for every combination of standard regularizers (diffusion, curvature, elastic, and fluid) and boundary conditions (Dirichlet, Neumann, and periodic).

1. Introduction
Any image registration algorithm relies on defining three components: an objective function indicating the similarity between two images, a class of geometric transformations relating the two images, and an optimization technique for optimizing the objective function over the space of valid transformations. Nonrigid registration algorithms define transformations as being parametric (e.g., local rigid, local affine, thin plate splines, B-splines, radial basis functions, and finite element method models) or nonparametric (diffeomorphic deformation fields). Modersitzki [11] provides an overview of nonparametric image registration methods and illustrates their flexibility.

In this paper, we focus on nonparametric image registration. Techniques for solving the nonparametric image registration problem typically define the objective function as a combination of two terms: a term that indicates the actual image similarity, and a term that regularizes the deformation field in order to ensure smoothness and prevent folds or tears. Popular similarity measures include the sum of squared differences (SSD) [9,11], cross correlation (CC) [9], correlation ratio (CR) [9] and mutual information (MI) [9]. The regularizers that have been presented in the literature are the diffusion [11], curvature [7,11], elastic [1,5,11], and fluid [2,5,11] regularizers.

Optimization of the objective function is achieved by solving a nonhomogeneous system of partial differential equations known as the Euler-Lagrange equations, which arise via the calculus of variations [8]. The variational (Gâteaux) derivatives of the regularizer and similarity measure yield, respectively, the partial differential operator and driving force that define the Euler-Lagrange equations.

Much of the research effort into nonparametric image registration over the last decade has gone into the development and use of numerical approaches to solve the Euler-Lagrange equations. Most techniques apply a fixed-point iteration scheme directly to the Euler-Lagrange equations, yielding an algorithm that requires successive solutions of linear PDE systems until a stationary solution is found. (Modersitzki [11] shows how an artificial time variable can be introduced to yield a modified set of equations, which can then also be solved via fixed-point iteration.) Early registration approaches [5] used successive overrelaxation (SOR) to solve the successive linear PDE systems arising from the use of elastic or fluid regularizers. SOR, like other iterative linear solvers, can be used with any boundary conditions but can be slow to converge (typically requiring $O(n^2)$ operations per fixed point iteration at the beginning of the fixed point iteration scheme, where $n$ is the total number of voxels; see [3]).

More recent techniques have been developed for specific combinations of regularizers and boundary conditions: the elastic or fluid regularizers with periodic [2,6,11,14] and homogeneous Dirichlet [4] boundary conditions, and the diffusion [11] and curvature [7,11] regularizers with homogeneous Neumann boundary conditions. However, none of these techniques have yet been extended or generalized to work for any combination of the standard regularizers and boundary conditions.
In this paper, we show that such a generalization is possible; namely, that Fourier based approaches can be developed that enable nonparametric image registration to be performed quickly for any combination of standard regularizers and boundary conditions. This is a direct consequence of the relationships between the Euler-Lagrange equations and the Poisson and nonhomogeneous biharmonic equations [15]. These relationships are easy to see for the diffusion and curvature regularizers, but they require a linear transformation of the Euler-Lagrange equations for the elastic and fluid regularizers. Exploiting these relationships yields fast registration approaches (requiring $O(n \log n)$ operations per fixed point iteration) that utilize the discrete sine transform (for homogeneous Dirichlet boundary conditions), the discrete cosine transform (for homogeneous Neumann boundary conditions), or the discrete Fourier transform (for periodic boundary conditions).

The remainder of this paper is organized as follows: Section 2 provides the mathematical background needed to understand the nonparametric image registration problem and current solution schemes. Section 3 describes Poisson’s equation and the nonhomogeneous biharmonic equation and shows how Fourier methods can be used to quickly solve these equations with respect to the three standard boundary conditions. Section 4 details how each of the standard regularizers relates to the Poisson and nonhomogeneous biharmonic equations, and thus, to their fast solutions. Finally, Section 5 illustrates a practical example of nonparametric image registration applied to mammography.

2. Nonparametric image registration

This section provides the mathematical preliminaries necessary to describe the nonparametric image registration problem. It also provides a summary of current solution techniques.

2.1. Problem formulation

The nonparametric image registration problem seeks to find the optimal deformation relating a reference image $I_{\text{ref}}$ and a floating image $I_{\text{float}}$, both defined on the closure of the open set $\Omega$. Deformation $\Phi$ is defined by:

$$\Phi(x) := x - u(x),$$

where $u(x)$ is the displacement, which is assumed to be diffeomorphic. In the sequel, we define $I_{\text{float}}^u$ as the deformed image $I_{\text{float}}(\Phi)$.

The optimal deformation is found by minimizing (or maximizing) an objective function $\mathcal{F}$ comprising a regularizing term $R$ and a (dis)similarity term $D$:

$$\min_u \mathcal{F}[u] := aR[u] + D[I_{\text{ref}}, I_{\text{float}}^u; u],$$

As shown in [11], this minimizer is characterized by the solution of the corresponding Euler-Lagrange equations:

$$aA[u(x)] - b(x, u(x)) = 0 \quad \forall x \in \Omega,$$

where $A$ and $b$ are related to the Gâteaux derivatives of $R$ and $D$, respectively.

2.1.1 Regularizers

The diffusion, curvature, and elastic regularizers and their corresponding partial differential operators are listed in Tables (1) and (2):

<table>
<thead>
<tr>
<th></th>
<th></th>
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</thead>
<tbody>
<tr>
<td>Diffusion</td>
<td>$\Delta u$</td>
</tr>
<tr>
<td>Curvature</td>
<td>$\Delta^2 u$</td>
</tr>
<tr>
<td>Elastic</td>
<td>$\mu \Delta u + (\lambda + \mu \nabla (\nabla \cdot u))$</td>
</tr>
<tr>
<td>Fluid</td>
<td>$\mu \Delta v + (\lambda + \mu \nabla (\nabla \cdot v))$</td>
</tr>
</tbody>
</table>

The symbols $\nabla$, $\nabla \cdot$, and $\Delta$ indicate the gradient, divergence, and Laplacian operators, respectively; $u_r$ is the $r^{th}$ component of the displacement field $u$, $\lambda$ and $\mu$ are the Lamé constants, and $\rho$ is the image dimension. Note that the fluid regularizer can be thought of as the elastic regularizer applied to the velocity field $v$, which is related to the displacement field $u$ by:

$$v(x; t) = \partial_t u(x; t) + (\nabla u(x; t))v(x; t).$$
2.1.2 Similarity terms

The force vector $\mathbf{b}$ relates to the Gâteaux derivative of the similarity term. For the sum of squared differences (SSD) similarity term, we have:

$$D_{\text{SSD}}[I_{\text{ref}}, I_{\text{float}}, \mathbf{u}] = \int_{\Omega} \left( I_{\text{float}}(x) - I_{\text{ref}}(x) \right)^2 \, dx$$

(5)

and

$$b_{\text{SSD}}(x, \mathbf{u}(x)) = \left( I_{\text{ref}}(x) - I_{\text{float}}(x) \right) \nabla I_{\text{float}}(x).$$

(6)

For the purposes of this paper, any of the standard similarity terms can be used. Therefore, without loss of generality, we assume that the SSD similarity term is used, and refer the interested reader to Chapter 5 of [9] for other similarity terms and their corresponding force vectors.

2.1.3 Boundary conditions

The Euler-Lagrange equations as shown in (3) are defined only on the interior of the image domain. In order to solve the Euler-Lagrange equations, boundary conditions must be specified. Three standard types of boundary conditions are used in this paper: homogeneous Dirichlet, homogeneous Neumann, and periodic.

Homogeneous Dirichlet boundary conditions state that the displacement field takes on the value of zero at the boundary:

$$\mathbf{u}(x) = 0 \quad \forall x \in \partial \Omega.$$  

(7)

Homogeneous Neumann boundary conditions state that the directional derivative of each component of the displacement field with respect to the normal vector at each boundary point is zero:

$$\langle \nabla \mathbf{u}(x), n(x) \rangle = 0 \quad \forall x \in \partial \Omega, \quad r = 1, \ldots, \rho.$$  

(8)

Finally, periodic boundary conditions assume that the image domain has been shifted and scaled to the unit hypercube; i.e., $\Omega = (0,1)^\rho$. These boundary conditions state:

$$u_r(x \mid x_r = 0) = u_r(x \mid x_r = 1) \quad \forall x \in \partial \Omega, \quad r, s = 1, \ldots, \rho.$$  

(9)

2.2. Current solution techniques

The Euler-Lagrange equations (3) are semi-linear, so a convenient way to solve them is by fixed-point iteration. Given an initial estimate $\mathbf{u}^{(0)}$ of the displacement (e.g., $\mathbf{u}^{(0)} = 0$), successive displacement estimates are implicitly defined by:

$$\alpha A \left[ \mathbf{u}^{(k+1)}(x) \right] = \mathbf{b}(x, \mathbf{u}^{(k)}(x)) \quad \forall x \in \Omega$$

(10)

Equation (10) is a linear PDE system that can be solved for successive values of $k$ until the iteration converges on a fixed displacement field $\mathbf{u}^{(k)}$.

A variety of techniques for numerically approximating the solution of (10) have arisen in the literature. All techniques have involved discretizing the PDE systems by making finite difference approximations to the gradient, divergence, and Laplacian operators. Early techniques [3] used iterative linear system solvers, such as successive overrelaxation, to solve the discrete version of (10).

A number of different authors [2,4,6,7,11,14] have recognized that (10) can be solved for specific choices of regularizer and/or boundary condition much more quickly than iterative linear system solvers would allow. With respect to the fluid regularizer, Modersitzki [11] and Bro-Nielsen and Gramkow [2] derive the eigenfunctions of the Navier-Lamé operator (corresponding to the elastic and fluid regularizers), and Modersitzki [11] shows how, under periodic boundary conditions, the knowledge of these eigenfunctions leads to a Fourier based approach to the solution of (10). Cahill et al. [4] extended this analysis to show that the discrete sine transform can be used if Dirichlet boundary conditions are considered. Other approaches based on inverse filtering [14] and multigrid techniques [6] have been applied when the fluid regularizer is used.

For the diffusion regularizer with homogeneous Neumann boundary conditions, Modersitzki [11] illustrates how additive operator splitting (AOS) technique yields a fast solution. He also shows that the discrete cosine transform can be used to solve (10) with the curvature regularizer and homogeneous Neumann boundary conditions.

Each of these techniques is applicable for a specific combination of regularizer and boundary conditions. In the remainder of this paper, we show that a class of transform-based techniques can be used to quickly solve (10) for any combination of the standard regularizers and boundary conditions.

3. Solving Poisson’s equation and the nonhomogeneous biharmonic equation

The PDE system in equation (10) can be directly related to either Poisson’s equation (for the diffusion regularizer) or the nonhomogeneous biharmonic equation (for the curvature, elastic, and fluid regularizers). Before we describe the specific relationships (in Section 4), we present both the Poisson equation and the nonhomogeneous biharmonic equation, and illustrate how transform-based techniques can be used to quickly solve
either equation when homogeneous Dirichlet, homogeneous Neumann, or periodic boundary conditions are chosen.

Poisson’s equation arises frequently in mechanical engineering, electrostatics, and physics. It is given by:

\[ \Delta y = f, \]  

(11)

where \( \Delta \) is the Laplacian operator, which is often denoted in the physics and engineering literature as \( \nabla^2 \). In physical problems, the Poisson equation is usually solved with respect to Dirichlet or Neumann boundary conditions.

The nonhomogeneous biharmonic equation also arises in mechanical engineering, specifically in linear elasticity theory. It is given by:

\[ \Delta^2 y = f, \]  

(12)

where \( \Delta^2 \) is sometimes referred to as the “double Laplacian” operator, and is often denoted as \( \nabla^4 \).

Numerous approaches exist for quickly solving (11) and (12); we focus on the Fourier methods originally presented in [13] because of their simplicity and elegance. The basic idea is to discretize the problem and then represent \( y \) as a linear combination of orthogonal basis functions that satisfy the boundary conditions and are eigenfunctions of the Laplacian operator.

We assume that \( y \) and \( f \) are sampled on an \( n \)-dimensional lattice that contains \( N_j \) samples along the \( j \)-th dimension:

\[ y = [y_k^j], \quad f = [f_k^j], \quad k_j = (k_{1j}, \ldots, k_{nj}), \quad j = 0, \ldots, N_j - 1. \]  

(13)

and that the discrete Laplacian operator \( \tilde{\Delta} \) is given by:

\[ \tilde{\Delta} y_k^j = \sum_{j=1}^{n} (y_{k+e_j}^j + y_{k-e_j}^j - 2y_k^j). \]  

(14)

where \( e_j \) is the \( j \)-th column of the \( n \times n \) identity matrix. Note that (14) is valid only for interior points; the definition of the discrete Laplacian must be modified on the boundaries to take into account the desired boundary conditions.

The discrete double Laplacian operator \( \tilde{\Delta}^2 \) can be determined by successively applying (14); i.e.,

\[ \left( \tilde{\Delta}^2 y \right)_k^j = \left( \tilde{\Delta} (\tilde{\Delta} y) \right)_k^j. \]  

(15)

Now suppose \( y \) and \( f \) are represented as linear combinations of orthogonal basis functions:

\[ y_j = \sum_k \hat{y}_k \theta_{j(k)} \forall j, \]  

(16)

\[ f_j = \sum_k \hat{f}_k \theta_{j(k)} \forall j, \]  

(17)

and that these basis functions each satisfy the boundary conditions. Furthermore, suppose the basis functions are eigenfunctions of the discrete Laplacian operator; i.e.,

\[ \tilde{\Delta} \theta_{j(k)} = \omega_k \theta_{j(k)}. \]  

(18)

Substituting (16)-(18) into (11) yields:

\[ \sum_k \hat{y}_k \omega_k \theta_{j(k)} = \sum_k \hat{f}_k \theta_{j(k)} \forall j. \]  

(19)

Since the basis functions are orthogonal, we can deduce that \( \hat{y}_k \omega_k = \hat{f}_k \); hence,

\[ \hat{y}_k = \frac{\hat{f}_k}{\omega_k} \forall k, \]  

(20)

so long as \( \omega_k \neq 0 \). (The only situations in which \( \omega_k = 0 \) occur when \( k = 0 \), as can be seen from the formulas for eigenvalues in the subsequent section. For Dirichlet boundary conditions, \( \theta_{j(0)} \) collapses and \( \hat{y}_0 \) can be assigned to zero. For Neumann or periodic boundary conditions, there is an existence/uniqueness problem due to the Fredholm alternative; however, in practice, assigning \( \hat{y}_0 = \hat{f}_0 \) appears to yield stable results.)

Therefore, we can state an elegant algorithm for solving the discretized version of (11):

**Algorithm (1): Solving the Poisson equation**

(i) Compute the coefficients of the expansion of \( f \) from (17);

(ii) Generate the values \( \hat{y}_k \) from (20); and,

(iii) Construct \( y \) from (16).

The nonhomogeneous biharmonic equation can be solved in much the same way by recognizing from (18) that the eigenvalues of the discrete double Laplacian operator are the squares of the eigenvalues of the discrete Laplacian operator:

\[ \tilde{\Delta}^2 \theta_{j(k)} = \omega_k^2 \theta_{j(k)}. \]  

(21)

Substituting (16), (17), and (21) into (12) yields:

\[ \sum_k \hat{y}_k \omega_k^2 \theta_{j(k)} = \sum_k \hat{f}_k \theta_{j(k)} \forall j. \]  

(22)

Therefore, in this case, we have:

\[ \hat{y}_k = \frac{\hat{f}_k}{\omega_k^2} \forall k. \]  

(23)
As in the case of solving the Poisson equation, we can choose \( \hat{y}_0 = 0 \) for Dirichlet boundary conditions, and \( \hat{y}_0 = \hat{f}_0 \) for Neumann or periodic boundary conditions.

Hence, we can state a similar algorithm for solving the discretized version of the nonhomogeneous biharmonic equation (12):

**Algorithm (2): Solving the nonhomogeneous biharmonic equation**

(i) Compute the coefficients of the expansion of \( f \) from (17);
(ii) Generate the values \( \hat{y}_k \) from (23); and,
(iii) Construct \( y \) from (16).

Now what remains is to identify a set of orthogonal basis functions that satisfy the boundary conditions, are eigenfunctions of the discrete Laplacian operator, and enable quick transformation of coordinates. The set of basis functions differ for each set of boundary conditions; therefore, we present them separately for each case in the following subsections.

### 3.1. Homogeneous Dirichlet boundary conditions

For the homogeneous Dirichlet boundary conditions, we present the following theorem:

**Theorem (1):** The orthogonal set of functions defined by:

\[
\theta_j^{(k)} = \beta_j \prod_{m=1}^{n} \sin \frac{k_j \pi}{N_{n-1}},
\]  

are eigenfunctions of the discrete Laplacian operator \( \tilde{\Delta} \) with corresponding eigenvalues:

\[
\omega_k = \sum_{m=1}^{n} \left( 2 \cos \frac{k_m \pi}{N_{n-1}} - 2 \right),
\]

and any function in their span satisfies the homogeneous Dirichlet boundary conditions.

**Proof:** The discrete Laplacian of (24) is given by:

\[
\tilde{\Delta} \theta_j^{(k)} = \beta_j \sum_{m=1}^{n} \left( \prod_{r=1}^{n} \sin \frac{k_j \pi}{N_{n-1}} \right) \left( \sin \frac{k_j \pi}{N_{n-1}} + \sin \frac{k_j \pi}{N_{n-1}} - 2 \sin \frac{k_j \pi}{N_{n-1}} \right)
\]

\[
= \beta_j \sum_{m=1}^{n} \left( \prod_{r=1}^{n} \sin \frac{k_j \pi}{N_{n-1}} \right) \left( \sin \left( \frac{k_j \pi}{N_{n-1}} \right) + \sin \left( \frac{k_j \pi}{N_{n-1}} + \frac{k_j \pi}{N_{n-1}} - 2 \sin \frac{k_j \pi}{N_{n-1}} \right) \right)
\]

(28)

Furthermore, each \( \theta_j^{(k)} = 0 \) if any \( j_m = 0 \) or \( j_m = N_m - 1 \). Therefore, any function in the span satisfies these conditions as well.

This choice of basis functions enables fast calculation of coefficients via the discrete sine transform (DST). (Reference [12] shows how the DST can be computed using the fast Fourier transform, or FFT.) Since the DST is its own inverse under this set of basis functions, we can restate the algorithms for solving Poisson’s equation and the nonhomogeneous biharmonic equation in the following way:

**Algorithm (3): Solving the Poisson equation (or the nonhomogeneous biharmonic equation) with homogeneous Dirichlet boundary conditions**

(i) Compute the DST of \( f \) to generate \( \hat{f} \);
(ii) Generate \( \hat{y} \) from (19) and (25) if solving the Poisson equation; from (23) and (25) if solving the nonhomogeneous biharmonic equation; and,
(iii) Compute the DST of \( \hat{y} \) to generate \( y \).

### 3.2. Homogeneous Neumann boundary conditions

For the homogeneous Neumann boundary conditions, we present the following theorem. The proof is omitted for space considerations, as it is similar to the proof of Theorem (1).

**Theorem (2):** The orthogonal set of functions defined by:

\[
\theta_j^{(k)} = \beta_j \prod_{m=1}^{n} \cos \frac{k_m \pi}{N_{n-1}}
\]

are eigenfunctions of the discrete Laplacian operator \( \tilde{\Delta} \) with corresponding eigenvalues:

\[
\omega_k = \sum_{m=1}^{n} \left( 2 \cos \frac{k_m \pi}{N_{n-1}} - 2 \right),
\]

(31)

and any function in their span satisfies the homogeneous Neumann boundary conditions.
This choice of basis functions enables fast calculation of coefficients via the discrete cosine transform (DCT). (Reference [12] shows how the DCT can be computed using the FFT.) Since the DCT is its own inverse under this set of basis functions, we can restate the algorithms for solving Poisson’s equation and the nonhomogeneous biharmonic equation in the following way:

**Algorithm (3): Solving the Poisson equation (or the nonhomogeneous biharmonic equation) with homogeneous Neumann boundary conditions**

(i) Compute the DCT of \( f \) to generate \( \hat{f} \);

(ii) Generate \( \hat{y} \) from (19) and (31) if solving the Poisson equation; from (23) and (31) if solving the nonhomogeneous biharmonic equation; and,

(iii) Compute the DCT of \( \hat{y} \) to generate \( y \).

### 3.3. Periodic boundary conditions

For the periodic boundary conditions, we present the following theorem. The proof is omitted for space considerations, as it is similar to the proof of Theorem (1).

**Theorem (3):** The orthogonal set of functions defined by:

\[
\theta_j^{(k)} = \beta_2 \prod_{m=1}^{n} \exp \left( \frac{2ikm \pi}{N_n} \right)
\]

are eigenfunctions of the discrete Laplacian operator \( \tilde{\Delta} \) with corresponding eigenvalues:

\[
\omega_k = \sum_{m=1}^{n} \left( 2 \cos \frac{2k \pi m}{N_n} - 2 \right),
\]

and any function in their span satisfies the periodic boundary conditions.

This choice of basis functions enables fast calculation of coefficients via the discrete Fourier transform (DFT). (The DFT can be computed using the FFT [12].) Under periodic boundary conditions, we can restate the algorithms for solving Poisson’s equation and the nonhomogeneous biharmonic equation in the following way:

**Algorithm (4): Solving the Poisson equation (or the nonhomogeneous biharmonic equation) with periodic boundary conditions**

(i) Compute the DFT of \( f \) to generate \( \hat{f} \);

(ii) Generate \( \hat{y} \) from (19) and (33) if solving the Poisson equation; from (23) and (33) if solving the nonhomogeneous biharmonic equation;

and,

(iii) Compute the inverse DFT of \( \hat{y} \) to generate \( y \).

### 4. Regularizers

In this section, we show that for any of the standard regularizers, the Euler-Lagrange equations (10) can be transformed into either a set of Poisson equations or a set of nonhomogeneous biharmonic equations. Therefore, depending on the choice of boundary condition, the Euler-Lagrange equations can be solved via DST, DCT, or DFT as shown in the previous section.

#### 4.1. Diffusion/Curvature

The diffusion partial differential operator, when substituted into (10), yields a Poisson equation for each component of the displacement field. Substitution of the curvature partial differential operator into (10) yields a nonhomogeneous biharmonic equation for each component of the displacement field. In either case, no transformation is necessary. Depending on the choice of boundary condition, one of the transform-based techniques shown in Section 3 can be used to solve for each displacement field component.

#### 4.2. Elastic/Fluid

The elastic and fluid partial differential operators, when substituted into (10), yield PDE systems that are not directly in the form of either Poisson’s equation or a nonhomogeneous biharmonic equation. However, this can be remedied by judiciously choosing a linear transformation that can be applied to both sides of (3).

**Theorem (4):** For the elastic and fluid regularizers, the PDE system (10), when premultiplied on both sides by the linear operator:

\[
L = (\lambda + 2\mu)\Delta I - (\lambda + \mu)\text{grad div}
\]

becomes a nonhomogeneous biharmonic equation.

**Proof:** It suffices to show that \( L[\mathbf{A}_{\text{elastic}}[\mathbf{u}]] \not= \Delta^2 \mathbf{u} \):

\[
L[\mathbf{A}_{\text{elastic}}[\mathbf{u}]] = (\lambda + 2\mu)\Delta[\mu\Delta \mathbf{u} + (\lambda + \mu)\nabla(\nabla \cdot \mathbf{u})] - (\lambda + \mu)\text{grad div}[\mu\Delta \mathbf{u} + (\lambda + \mu)\nabla(\nabla \cdot \mathbf{u})] \]

\[
= \mu(\lambda + 2\mu)\nabla^2 \mathbf{u} + (\lambda + \mu)(\lambda + \mu)\nabla(\nabla \cdot \mathbf{u}) - (\lambda + \mu)^2 \nabla(\nabla \cdot \mathbf{u})
\]

\[
= \mu(\lambda + 2\mu)\nabla^2 \mathbf{u} + (\lambda + \mu)(\lambda + \mu)\nabla(\nabla \cdot \mathbf{u}) - (\lambda + \mu)^2 \Delta \nabla(\nabla \cdot \mathbf{u})
\]

(35)
\[ = \mu(\lambda + 2\mu)\Delta u. \quad \Box \] (38)

Note that this proof is independent of dimension. (In two dimensions, it happens that \(L\) is simply the adjoint of the Navier-Lamé operator. This is not true, however, in higher dimensions.)

Hence, depending on the choice of boundary condition, nonparametric image registration with an elastic or fluid regularizer can be performed using one of the transform-based techniques shown in Section 3, provided that the force field is premultiplied by (34) and divided by the coefficient of (38).

5. Breast image registration example

In order to illustrate the results of nonparametric image registration subject to any combination of the standard regularizers and boundary conditions, we consider a common application: registration of 3-D magnetic resonance images of the breast. To compare the various regularizers and boundary conditions in a way that provides some degree of quantitative validation, we use two 3-D breast images that are related by a known deformation and have been provided by the authors of [10].

A FEM model is applied to a standard 3-D MR breast image in order to simulate two 3-D images of the breast, corresponding to different rotations and plate compressions of a baseline 3-D MR image that mimic the type of conditions under which X-ray mammograms are captured. The reference 3-D image has been constructed using no rotation and a plate compression of 60%; the floating 3-D image has been constructed using a rotation of 10° and a plate compression of 70%. The use of the FEM model enables ground truth displacements to be computed for each voxel in the floating image.

The 3-D images are isotropically sampled and contain 160×160×92 voxels, each with side length 1.3672mm. Registration of the floating image to the reference image is performed using every combination of regularizer and boundary conditions. In all cases, the force vector (6) corresponding to the SSD similarity term (5) is used. The fixed point iteration is terminated when the maximum displacement in a given iteration is less than 1/4 the length of a side of a voxel, or, alternatively, when the reduction in similarity measure is less than 0.01%.

Figure (2) shows orthogonal slices through the floating image, along with yellow cones that indicate local behavior of the deformation found using the diffusion regularizer and homogeneous Dirichlet boundary conditions.

Figure (3) shows axial, coronal and sagittal slices of the difference images before and after registration of the 3-D MR breast images. The difference image on the left is computed from the original floating image and the reference image, and the difference image on the right is computed after the floating image has been warped into the same frame of reference as the reference image using the deformation illustrated in Figure (2).

To compare the various combinations of regularizer and boundary conditions, we investigate three measures: the number of iterations required for the fixed point iteration (10) to converge to a solution; the ratio of the SSD of the registered images to the SSD of the unregistered images; and, the target registration error (TRE), reported as the root median square (RMedS) error in millimeters between predicted displacements and ground truth displacements.
over all voxels within the breast in the floating image. (Note that the RMedS error of the unregistered images is 3.7026mm). These measures are reported in the following tables.

<table>
<thead>
<tr>
<th></th>
<th>Dirichlet</th>
<th>Neumann</th>
<th>Periodic</th>
</tr>
</thead>
<tbody>
<tr>
<td>Diffusion</td>
<td>50</td>
<td>51</td>
<td>16</td>
</tr>
<tr>
<td>Curvature</td>
<td>58</td>
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<td>34</td>
</tr>
<tr>
<td>Elastic</td>
<td>56</td>
<td>60</td>
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</tr>
<tr>
<td>Fluid</td>
<td>55</td>
<td>60</td>
<td>21</td>
</tr>
</tbody>
</table>

Table (3): Number of iterations required for the fixed-point iteration to converge to a solution.

<table>
<thead>
<tr>
<th></th>
<th>Dirichlet</th>
<th>Neumann</th>
<th>Periodic</th>
</tr>
</thead>
<tbody>
<tr>
<td>Diffusion</td>
<td>0.0492</td>
<td>0.1205</td>
<td>0.5622</td>
</tr>
<tr>
<td>Curvature</td>
<td>0.1819</td>
<td>0.3709</td>
<td>0.5887</td>
</tr>
<tr>
<td>Elastic</td>
<td>0.0525</td>
<td>0.1242</td>
<td>0.5551</td>
</tr>
<tr>
<td>Fluid</td>
<td>0.0521</td>
<td>0.1236</td>
<td>0.5511</td>
</tr>
</tbody>
</table>

Table (4): Ratio of SSD of registered images to SSD of unregistered images.

<table>
<thead>
<tr>
<th></th>
<th>Dirichlet</th>
<th>Neumann</th>
<th>Periodic</th>
</tr>
</thead>
<tbody>
<tr>
<td>Diffusion</td>
<td>0.7039</td>
<td>1.0410</td>
<td>3.0352</td>
</tr>
<tr>
<td>Curvature</td>
<td>1.1069</td>
<td>1.7576</td>
<td>2.7896</td>
</tr>
<tr>
<td>Elastic</td>
<td>0.6532</td>
<td>0.9555</td>
<td>3.0352</td>
</tr>
<tr>
<td>Fluid</td>
<td>0.6549</td>
<td>0.9557</td>
<td>3.0276</td>
</tr>
</tbody>
</table>

Table (5): Target Registration Error (TRE): RMedS error (in mm) between predicted displacements and ground truth over all voxels in the breast.

As can be seen from Table (5), the target registration error appears to be best for homogeneous Dirichlet boundary conditions and worst for periodic. The elastic and fluid regularizers also appear to be the best performers, with the diffusion regularizer close behind.

It is important not to draw sweeping conclusions from this single registration example; in order to generalize, it would be useful to analyze registrations from a number of examples across a variety of applications. Rather, this single example is meant to provide a practical look at how the Fourier methods for nonparametric image registration can be applied.

Acknowledgements

N. D. Cahill would like to the following members of the Centre for Medical Image Computing at the University College London: Bill Crum, for helpful discussions on fluid registration, and John Hipwell and Christine Tanner, for providing the 3D breast images.

References