Fractional Acquisition in Graphs

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June 9, 2014

Abstract

Let $G$ be a vertex-weighted graph in which each vertex has weight 1. Given a vertex $u$ with positive weight and a neighbor $v$ whose weight is at least the weight on $u$, a fractional acquisition move transfers some amount of weight at $u$ from $u$ to $v$. The fractional acquisition number of $G$, written $a_f(G)$, is the minimum number of vertices with positive weight after a sequence of fractional acquisition moves in $G$. In this paper, we determine the fractional acquisition number of all graphs: if $G$ is an $n$-vertex path or cycle, then $a_f(G) = \lceil n/4 \rceil$; if $G$ is connected with maximum degree at least 3, then $a_f(G) = 1$.

Keywords: acquisition, graph acquisition, fractional acquisition

1 Introduction

Consider an army that is deployed at a collection of bases, some of which are connected by roads. We wish to consolidate all of the troops at a single base. Troops are allowed to move only to neighboring bases with at least as many troops as their current base. Is it possible for all of the troops to reach a single base?

Let $G$ be a vertex-weighted graph and let $w : V(G) \to \mathbb{R}_{\geq 0}$ be the weight assignment on $G$. In a weighted graph $G$, a fractional acquisition move transfers some positive amount of weight from a vertex $u$ to a neighbor $v$, provided that the weight on $v$ is at least the weight on $u$. The amount of weight transferred cannot exceed the weight on $u$. We refer to a succession of fractional acquisition moves as a fractional protocol; throughout this paper we assume that all protocols are finite in length. We study fractional protocols on graphs in which every vertex starts with weight 1. The fractional acquisition number of $G$, denoted $a_f(G)$, is the minimum size of the set of vertices with positive weight after a fractional protocol on $G$. The weight assignment $w$ is feasible if there is a fractional protocol that achieves $w$ starting from the initial all-1s weight assignment.

Previous work on acquisition in graphs has focused on acquisition moves that transfer all of the weight from a vertex to its neighbor with at least the same weight; we call such an acquisition...
move a total acquisition move. Analogously, the minimum number of vertices in a graph $G$ with positive weight after a sequence of total acquisition moves is the total acquisition number of $G$, denoted $a_t(G)$. Lampert and Slater [1] proved that for $n \geq 2$, if $G$ is a connected $n$-vertex graph, then $a_t(G) \leq (n + 1)/3$. They also provided a lower bound on the total acquisition number of a connected graph depending on its degree sequence. LeSaulnier and West [3] characterized all graphs achieving equality for the total acquisition upper bound. Slater and Wang [4] proved that determining if the total acquisition number of a given graph $G$ is 1 is an NP-complete problem and provided a linear time algorithm that determines the total acquisition number for caterpillars. LeSaulnier et al. [2] then provided a polynomial time algorithm to test $a_t(T) \leq k$ where $T$ is a tree and $k$ is any fixed positive integer. They also established numerous sufficient conditions for a graph to have total acquisition number 1.

In [1], Lampert and Slater used the term consolidation to refer to an acquisition move that moves an integer amount of weight from a vertex to its neighbor. Clearly, each total acquisition move is also a consolidation. A fractional acquisition move is a further generalization of a consolidation. Because each total acquisition move is also a fractional acquisition move, we have that $a_f(G) \leq a_t(G)$ for all $G$.

In this paper we determine the fractional acquisition number of all graphs. In Section 2, we prove that if $G$ is a connected graph and $\Delta(G) \geq 3$, then $a_f(G) = 1$. This is a startling contrast to what is known about total acquisition numbers. All sufficient conditions in [2] for a graph to have total acquisition number 1 depend on dominating cliques or vertices of large degree whose neighborhoods are dominating sets. Such graphs have low diameter and strong structural requirements. More generally, it is conjectured in [2] that if $G$ is an $n$-vertex graph with diameter 2, then $a_t(G)$ is bounded by an absolute constant (perhaps even 2), but the best known bound is that $a_t(G) \leq 32 \log n \log \log n$. In contrast, adding a single pendant to a vertex of degree 2 in an $(n - 1)$-vertex path yields an $n$-vertex graph with maximum degree 3, diameter $n - 2$, and fractional acquisition number 1. We prove that trees with maximum degree at least 3 have fractional acquisition number 1 by inductively constructing a protocol that yields a weight distribution with certain desired properties. For arbitrary connected graphs with maximum degree at least 3 it is then sufficient to use the edge set of a spanning tree containing a vertex of degree at least 3.

In Section 3, we prove that the fractional acquisition numbers of the $n$-vertex path and the $n$-vertex cycle are $[n/4]$. Combined with the result on connected graphs with maximum degree at least 3 this determines the fractional acquisition number of every connected graph, and consequently all graphs.

Interestingly, when $G$ is a path or cycle, the fractional acquisition number and total acquisition number of $G$ are equal (the total acquisition number is determined in [1]). Thus the freedom of fractional acquisition moves does not provide an advantage when the maximum degree of a graph
is 2. The proof that \( a_t(P_n) = a_t(C_n) = \lceil n/4 \rceil \) follows from the observations that each total acquisition move at most doubles the weight at a vertex and each edge can be used at most once by total acquisition moves (\( P_n \) and \( C_n \) denote the \( n \)-vertex path and cycle, respectively). Therefore the maximum amount of weight that a vertex in a path or cycle can acquire via total acquisition moves is 4, and the result follows. In contrast, in Section 3 we show that the maximum amount of weight that a vertex in \( P_n \) may acquire via fractional acquisition moves grows with \( n \).

Though we determine the fractional acquisition number of all graphs, many interesting open questions remain, particularly with respect to the efficiency of fractional protocols. We present open questions and conjectures in Section 4.

For any undefined terminology, we refer the reader to [5].

## 2 General Graphs

We determine the fractional acquisition number of graphs with maximum degree at least 3. If \( u \) and \( v \) are vertices in a tree \( T \), we let \( T(u, v) \) denote the unique \( u, v \)-path in \( T \). Also, we refer to the vertices in a tree \( T \) with degree at least 3 as branch vertices.

**Theorem 1.** If \( G \) is a connected graph with \( \Delta(G) \geq 3 \), then \( a_f(G) = 1 \).

In the proof of Theorem 1, we make extensive use of paths with weight assignments that allow all of the weight on the path to be acquired by a single vertex. An ascending path is a path \( v_1v_2\ldots v_k \) with a weight assignment \( w \) such that \( w(v_1) \leq w(v_2) \) and \( w(v_i) < (v_{i+1}) \) for \( i \in \{2, \ldots, k-1 \} \). When it is convenient, we will say that such a path \( P \) ascends to \( v_k \), or that \( P \) is \( v_k \)-ascending. An ascending path \( P \) is strictly ascending if \( w(v_1) < w(v_2) \). A weighted tree \( T \) is ascending if there is a vertex \( v \in V(T) \) such that for every vertex \( u \) in the tree, \( T(u, v) \) is \( v \)-ascending.

We will frequently use a protocol that moves weight along an ascending path. Let \( P = v_1 \ldots v_k \) be a \( k \)-vertex path with a positive weight assignment \( w \) that ascends from \( v_1 \) to \( v_k \). Let \( c = \min(\{w(v_1)\} \cup \{w(v_{i+1}) - w(v_i) : 2 \leq i \leq k-1 \}) \). Define the path protocol, denoted \( A(v_1, v_k) \), as follows. Transfer weight \( c \) from \( v_1 \) to \( v_k \) while moving no other weight; let step \( i \) in the protocol move weight \( c \) from \( v_i \) to \( v_{i+1} \). After the \( i \)th step, the new weight on \( v_{i+1} \) is \( w(v_{i+1}) + c \); since \( w(v_{i+2}) \geq w(v_{i+1}) + c \), the protocol can continue. On step \( k-1 \), the “packet” of weight \( c \) reaches \( v_k \) and the protocol terminates. We denote \( \ell \) repeated applications of the path protocol \( A(v_1, v_n) \) by \( A(v_1, v_n) \ell \).

**Lemma 2.** If a tree \( T \) has a feasible weight assignment \( w \) that ascends to a vertex \( r \), then \( a_f(T) = 1 \) and \( r \) can acquire all of the weight in \( T \).
Proof. We use induction on the number of vertices in $T$ with positive weight. If $r$ is the only vertex with positive weight, then $a_f(T) = 1$. Otherwise, let $u$ be a vertex with positive weight that is farthest from $r$. Use the path protocol to move weight to $u$ from $r$; let $c$ be the amount of weight transferred from $u$ to $r$. Applying the protocol $A(u, r)^{\lceil w(u)/c \rceil}$ leaves $u$ with weight $0$, $r$ with weight $w(r) + w(u)$, and all other weights unchanged. Thus we have decreased the number of vertices with positive weight and we apply the induction hypothesis.

To prove Theorem 1, we need only fractional acquisition moves that transfer rational amounts of weight; therefore we introduce a new model of fractional acquisition, which we call the normalized model. Let each vertex start with weight $0$, and move finite positive amounts of weight, allowing negative weights on vertices. As with fractional acquisition moves, moving weight from $u$ to $v$ is valid only if the weight on $v$ is at least the weight on $u$. A protocol of such moves is a normalized protocol; all normalized protocols we use are finite in length. In the normalized model, the vertex weights always sum to $0$. A weight distribution where the weights sum to $0$ is called a normalized weight distribution.

For the proof of Theorem 1, it suffices to consider only normalized acquisition moves that transfer integer amounts of weight. When convenient, we will refer to the units of weight that are moving around the graph as chips. When working in the normalized model, we will assume that the path protocol $A(u, v)$ always transfers weight $1$ from $u$ to $v$. For any normalized protocol $A$ we obtain a corresponding fractional protocol $A'$.

**Lemma 3.** If $A$ is a normalized protocol on a graph $G$, then there is a corresponding fractional protocol $A'$ on $G$ such that a path in $G$ is ascending after protocol $A$ if and only if it is ascending after protocol $A'$.

**Proof.** Let $G$ be a weighted graph in which every vertex has weight $0$. Let $A$ be a normalized protocol consisting of $k$ normalized acquisition moves, and let $w_t(v)$ be the weight on $v$ after the first $t$ moves of $A$. Observe that $\min_{v \in V(G)} w_t(v)$ is nonincreasing in $t$. Let $m = \min_{v \in V(G)} w_k(v)$, that is, the largest negative weight that any vertex attains during the protocol $A$.

If step $t$ in $A$ moves weight $a$ from $u$ to $v$, then step $t$ in $A'$ moves weight $a/(|m| + 1)$ from $u$ to $v$. Let $w'_t$ be the weight distribution on $G$ (starting from the all-1s distribution) after the first $t$ moves of $A'$. Thus, for each $v \in V(G)$ and each $t \in \{0, \ldots, k\}$, we have $w'_t(v) = 1 + \frac{1}{|m|+1} w_t(v)$. Through the course of $A'$, all vertices have positive weight, and every move is a valid fractional acquisition move. Since an increasing linear function has been applied to the weights obtained from protocol $A$, a path in $G$ is ascending after protocol $A$ if and only if it is ascending after protocol $A'$.

We now prove a normalized acquisition analogue of Lemma 2. The definition of an ascending weight assignment extends to negative values.
Lemma 4. If \( G \) is a graph and a normalized protocol \( \mathcal{A} \) yields an ascending weight assignment on a spanning tree of \( G \), then \( a_f(G) = 1 \).

Proof. By Lemma 3, there is a fractional protocol \( \mathcal{A}' \) that produces an ascending weight assignment \( w \) on a spanning tree \( T \) of \( G \); let \( v \) be the vertex such that \( w \) is \( v \)-ascending. By Lemma 2, it is possible to transfer all of the weight in \( T \) (hence \( G \)) to \( v \). Thus \( a_f(G) = 1 \).

In light of Lemma 4, it is sufficient to show that on any graph \( G \) satisfying \( \Delta(G) \geq 3 \) there is a normalized protocol \( \mathcal{A} \) that produces an ascending weight assignment on a spanning tree of \( G \). Note that in the normalized model, the path protocol can also be used when vertices in a path have negative weight. All remaining protocols and weight distributions in this section are normalized.

Lemma 5. Let \( T \) be a weighted tree, and let \( v \) and \( u \) be vertices in \( T \) such that \( u \) is not a leaf. Starting from a \( v \)-ascending normalized weight distribution \( w \), there is a normalized protocol that produces a \( u \)-ascending normalized weight distribution.

Proof. Let \( v' \) be the neighbor of \( v \) in \( T(v, u) \), and let \( w(v) - w(v') = a \). It suffices to show that we can produce a \( v' \)-ascending weight assignment. Let \( x \) be a leaf in \( T \) such that \( u \) lies on the unique \( x, v \)-path \( T(x, v) \); the weights on that path ascend to \( v \). Because we are allowed to drive the weight of \( x \) negative, we can apply the protocol \( \mathcal{A}(x, v')^{a+1} \), producing a \( v' \)-ascending weight assignment.

We now present a protocol that, in conjunction with Lemma 5, yields a weight assignment on any subdivision of \( K_{1,3} \) that ascends to the branch vertex. We make frequent use of two protocols on strictly ascending paths. Suppose that the path \( v_1 \ldots v_m \) ascends strictly to \( v_m \). Let \( \overline{\mathcal{A}}(v_1, v_m) \) denote the protocol that for \( i \) from 1 to \( m - 1 \) (in order) moves one chip from \( v_i \) to \( v_m \). Thus \( \overline{\mathcal{A}}(v_1, v_m) \) has the effect of increasing the weight on \( v_m \) by \( m - 1 \) and decreasing the weight on \( v_i \) by 1 for \( i \in \{1, \ldots, m-1\} \). Let \( \overline{\mathcal{A}}(v_1, v_m) \) denote the protocol that for \( i \) from \( m \) to 2 (in order) moves one chip from \( v_1 \) to \( v_i \). Thus \( \overline{\mathcal{A}}(v_1, v_m) \) has the effect of decreasing the weight on \( v_1 \) by \( m - 1 \) and increasing the weight on \( v_i \) by 1 for \( i \in \{2, \ldots, m\} \).

Lemma 6. Let \( T \) be a subdivision of \( K_{1,3} \) with branch vertex \( v \) and leaves \( z, u, \) and \( u' \). Let \( v_0, v_1, \ldots, v_{m-1}, v_m \) be the vertices of the \( z, v \)-path in \( T \) with \( v_0 = z \) and \( v_m = v \). Let \( w \) be a normalized weight distribution on \( T \). If \( w(v_i) = 0 \) for \( i \in \{1, \ldots, m\} \), \( w(v_0) \geq 0 \), and \( T(u, u') \) ascends strictly to \( u' \) under \( w \), then there is normalized protocol on \( T \) that produces a \( u' \)-ascending weight assignment.

Proof. We describe \( m - 1 \) protocols \( \mathcal{A}_1, \ldots, \mathcal{A}_{m-1} \) that, when performed successively, result in the desired weight assignment. We will prove by induction on \( k \) that after \( \mathcal{A}_k \), \( T(u, u') \) is strictly ascending, \( T(v_0, v_k) \) is ascending, and \( w(v_i) = w(v_{i+1}) \) whenever \( i - k \) is positive and odd. This
process is illustrated in Figure 1; note that $u'$ and $u$ are not necessarily neighbors of $v$. The initial weight assignment is produced by a null protocol that we call $A_0$, which has the required properties for $k = 0$.

For $k \geq 1$, let $w(v_{k-1}) = a$ after $A_{k-1}$. The first move in $A_k$ transfers weight from $v_{k+1}$ to $v_k$ to produce $w(v_k) = a + 1$. This is possible because $v_{k+1}$ and $v_k$ have the same weight. For $j \in \{2, \ldots, \lceil (m - k)/2 \rceil \}$, the $j$th move in $A_k$ moves weight on the edge $v_{k+2j-1}v_{k+2j-2}$ so that $w(v_{k+2j-2}) = w(v_{k+2j-3})$. When $m - (k - 1)$ is even, weight must move across the edge $v_mv_{m-1}$. If weight $b$ moves from $v_{m-1}$ to $v_m$, we instead apply $A(v_{m-1}, u')^b$. If weight $b$ moves from $v_m$ to $v_{m-1}$, first apply $A(u, u')$ to reduce $w(v_m)$ by 1 and then apply $A(u, v_{m-1})^b$. When $m - (k - 1)$ is odd, no weight moves along the edge $v_mv_{m-1}$, however we wish to ensure that $w(v_m) = w(v_{m-1})$. If $w(v_m) > w(v_{m-1})$, then apply $A(u, u')^{w(v_m) - w(v_{m-1})}$. If $w(v_m) < w(v_{m-1})$, then apply $A(u, u')^{w(v_{m-1}) - w(v_m)}$. After $A_k$, the ascending path in $T(z, v)$ has length $k$.

After $A_{m-1}$, there is an ascending path of length $m - 1$ in $T(z, v)$. Therefore $T(z, v)$ ascends to $v$, and $T$ has the desired $u'$-ascending weight assignment.

Figure 1: A subdivision of $K_{1,3}$ with the relative weight of each vertex represented by its vertical position. The final weight assignment ascends to $u'$.

We now define a generalization of the path protocol that applies to trees. If $T$ is a tree with a weight assignment that ascends to $v$, the tree protocol, denoted $\overline{A}(T, v)$, consists of applying the path protocol $A(u, v)$ once for each $u$ in $T$, ordered so that if $u'$ is in $T(u, v)$, then $A(u, v)$ occurs before $A(u', v)$. Thus weight 1 moves from each vertex in $T$ to $v$.

**Lemma 7.** If $T$ is a subdivided star with at least three leaves and branch vertex $v$, then there is a normalized protocol that produces a $v$-ascending weight assignment.

**Proof.** If $T$ is a star, then the result holds using one move that transfers weight from a leaf to $v$. Thus we may assume that at least one path emanating from $v$ has length at least 2. We use induction on the degree of $v$. Let $u_1, \ldots, u_k$ denote the leaves in $T$, and let $v_i$ denote the neighbor of $v$ in $T(v, u_i)$.

For the basis of the induction, consider the case $k = 3$. By symmetry, we may assume that $u_3$ is not adjacent to $v$. Let $T'$ be the subtree of $T$ consisting of $T(u_3, v)$ and the edges $vv_1$ and $vv_2$.

Move weight 1 from $v$ to $v_1$ and from $v_3$ to its neighbor $z$ in $T(v_3, u_3)$. Move weight 1 from $v_3$ to $v$ and then apply $A(v_2, v_1)^3$, yielding $w(v, v_1, v_2, v_3, z) = (0, 4, -3, -2, 1)$ (Figure 2 (B)).
Move weight 3 from v to v₁ and apply \( A(v₂, v₃) \); now \( w(v, v₁, v₂, v₃, z) = (-3, 7, -6, 1, 1) \) (Figure 2 (C)). Move weight 5 from v₂ to v, and then apply \( A(z, v₁) \); now \( w(v, v₁, v₂, v₃, z) = (2, 8, -11, 1, 0) \) (Figure 2 (D)). Finally, apply \( A(v₃, v₁) \), and then move weight 2 from v to v₁; now \( w(v, v₁, v₂, v₃, z) = (0, 11, -11, 0, 0) \) (Figure 2 (E)).

By Lemma 6, there is now a protocol that we can apply to \( T' \) so that \( T(u₃, v) \) ascends to v and \( w(v₁) > w(v) > w(v₂) \). Apply the protocol \( \mathcal{A}(u₃, v₁)^{w(v)−w(v₂)+1} \) so that the weights on \( T(u₃, v₁) \) ascend to v₁ and \( w(v) < w(v₂) \). Now apply the repeated path protocol \( \mathcal{A}(u₃, v₂)^{w(v₂)} \) so that \( w(v₂) = 0 \), then apply \( \mathcal{A}(u₃, v₃)^{w(v)} \) so that \( w(v) = 0 \). Now \( w(v₃) < 0 \), and the weights on the \( u₃, v₁ \)-path ascend to v₁.

Applying the protocol of Lemma 6 with \( (v₁, u₂, u₃) \) as \( (u', z, u) \), respectively, now produces a weight assignment on \( T \) such that \( T(u₃, v₁) \) and \( T(u₂, v₁) \) are ascending. We then apply the protocol \( \mathcal{A}(u₃, v)^{w(v₁)−w(v₂)+1} \) and then \( \mathcal{A}(u₃, v₃)^{w(v₁)−w(v₃)+2} \) to ensure \( w(v₃) > w(v) > w(v₁) \). We then apply \( \mathcal{A}(v₁, v₃)^{w(v₁)} \) and \( \mathcal{A}(u₂, v₃)^{w(v)} \) so that \( w(v₁) = w(v) = 0 \) and \( T(u₂, v) \) ascends to v. Applying Lemma 6 again with \( (v₃, u₁, u₂) \) as \( (u', z, u) \) yields a weight assignment on \( T \) that ascends to v₃. Lemma 5 now produces a v-ascending weight assignment, completing the base case.

Now assume that \( m \geq 4 \) and \( T \) is a subdivision of \( K_{1,m} \) with branch vertex v and leaves \( u₁, \ldots, uₘ \). Let \( v_i \) be the neighbor of v in \( T(uᵢ, v) \) for \( i \in \{1, \ldots, m\} \). Because \( T \) is not a star, we may assume that \( u₁ \neq v₁ \). Let B be the vertex set of \( T(vᵢ, uᵢ) \). By the induction hypothesis, we can produce a weight assignment on \( T - B \) that ascends to v. Because \( u₁ \neq v₁ \), Lemma 5 implies that there is a protocol that produces a \( v₁ \)-ascending weight assignment \( w \) on \( T - B \). If \( w(v) > 0 \), then \( \mathcal{A}(T - B, v₁)^{w(v)} \) produces a \( v₁ \)-ascending weight assignment on \( T - B \) in which \( w(v) = 0 \). If \( w(v) < 0 \), then \( \mathcal{A}(u₂, v₁)^{w(v)} \) produces a \( v₁ \)-ascending weight assignment on \( T - B \) in which \( w(v) = 0 \). We now apply Lemma 6 to the subtree of \( T \) consisting of the paths joining v to \( uᵢ \) and \( u₂ \) along with the edge \( vv₁ \) to attain a \( v₁ \)-ascending weight assignment. Lemma 5 then yields a weight assignment on \( T \) that ascends to v.

Lemma 7 will serve as the base case of an inductive proof of Theorem 1.
Proof of Theorem 1. If $G$ has a vertex with degree at least 3, then $G$ has a spanning tree $T$ that has a vertex with degree at least 3. We show that if $T$ is a tree satisfying $\Delta(T) \geq 3$ in which every vertex has weight 0, then there is a normalized protocol that produces an ascending weight assignment on $T$. We use induction on the number of branch vertices in $T$. If $T$ has one branch vertex, then $T$ is a subdivided star and Lemma 7 applies. Thus we suppose that $T$ has $k$ branch vertices with $k \geq 2$.

If $T$ contains a branch vertex $x$ such that $x$ is adjacent to at least $d(x) - 2$ leaves, then consider the tree $T'$ obtained by deleting $d(x) - 2$ leaves adjacent to $x$. As $T'$ has $k - 1$ branch vertices, there is a normalized protocol that produces an ascending weight assignment on $T'$. Because $x$ is not a leaf in $T'$, Lemma 5 implies that there is a feasible $x$-ascending weight assignment on $T'$. In such an assignment, $w(x) > 0$, and as the leaves that were deleted from $T$ to form $T'$ still have weight 0 and are all adjacent to $x$, this protocol also produces an ascending weight assignment on $T$.

Thus we may assume that each branch vertex in $T$ has at least three neighbors that are not leaves. Let $x$ and $y$ be two branch vertices in $T$, let $x'$ be the neighbor of $x$ in $T(x, y)$, and let $v$ and $v'$ be two additional neighbors of $x$ that are not leaves. Consider the components of $T - x$. Let $T_1$ be the component containing $x'$, let $T_2$ be the component containing $v'$, let $T_3$ be the component containing $v$.

Let $T' = T - (T_1 - x')$. Because $T'$ has fewer than $k$ branch vertices and $v$ is not a leaf, the induction hypothesis and Lemma 5 imply that there is a feasible $v$-ascending weight assignment on $T'$. We now wish to make additional normalized acquisition moves so that $w(x) = w(x') = w(v') = 0$. First, repeatedly apply $\mathcal{A}(T', v)$ until $w(x) < 0$; note that this implies that $w(x') < 0$. Then apply $\mathcal{A}(T' - x', v)$ until $w(x') < w(x')$. If $u$ is a leaf in $T_2$, then $v'$ and $x$ lie in $T(u, x')$ and $T(u, x')$ is now $x'$-ascending. Thus we may apply $\mathcal{A}(u, x')^{|w(x')|}$, $\mathcal{A}(u, x)^{|w(x)|}$, and then $\mathcal{A}(u, v')^{|w(v')|}$ to obtain a weight assignment in which $w(x) = w(x') = w(v') = 0$. Note that $T_2$ is $v'$-ascending, and that $v$ has the maximum weight in $T'$.

Let $T'' = T_1 + x + v'$. All of the vertices in $T''$ have weight 0 and $T''$ has fewer than $k$ branch vertices, so there is a feasible ascending weight assignment on $T''$. Furthermore, as $x$ is not a leaf in $T''$, by Lemma 5 there is a feasible $x$-ascending weight assignment on $T''$. Combining this weight assignment with the weight assignment on $T'$ produces a weight assignment on $T$ that is either $x$- or $v$-ascending with the exception of the vertices in $T_2$. Since there is a leaf $u'$ in $T_3$, it is possible to apply the path protocol $\mathcal{A}(u', v)$ repeatedly so that $w(v) > w(x)$. Note that $T - T_2$ is $v$-ascending. At this point, application of $\mathcal{A}(T - T_2, v)^{w(x) - w(v') + 1}$ ensures that $w(x) < w(v')$. Then, using a leaf $z$ in $T_1$, the protocols $\mathcal{A}(z, v')^{w(x) - w(v') + 1}$ and $\mathcal{A}(z, x)^{w(v) - w(x) + 2}$ produce a weight assignment satisfying $w(x) > w(v') > w(v)$. The resulting weight assignment is $x$-ascending.

Since there is a normalized protocol on a spanning tree of $G$ that yields an ascending weight
assignment, Lemma 4 implies that \( a_f(G) = 1 \).

### 3 Paths and Cycles

In this section, all protocols are fractional acquisition protocols, and negative weights are not allowed. To determine the fractional acquisition number of paths and cycles, we model the weight on a vertex \( v \) as a vertical bar with length \( w(v) \). When \( v \) acquires weight \( a \) from \( u \), cut an interval of length \( a \) off of the top of the bar at \( u \) and attach it to the top of the interval at \( v \). All vertices start with weight 1, and we define the top of each initial bar to be a tip. Throughout the course of a fractional acquisition protocol, define the height of a particular tip to be the length of the longest (sub)bar of which it is the top. In this model of fractional acquisition, the height of a tip is nondecreasing. Figure 3 provides an example of this model.

**Theorem 8.** If \( G \) is the \( n \)-vertex path, then \( a_f(G) = \lceil n/4 \rceil \).

**Proof.** In [1], Lampert and Slater showed that \( a_t(P_n) = \lceil n/4 \rceil \), so we conclude that \( a_f(P_n) \leq \lceil n/4 \rceil \).

Let \( v_1, \ldots, v_5 \), be any five vertices in \( P_n \) labeled in order. Because the height of a tip increases each time it is moved, it is not possible for a fractional acquisition move to transfer a tip to a vertex with weight less than 1. The first fractional acquisition move involving \( v_3 \) results in either \( v_3 \) or one of its neighbors having weight less than 1. The vertex with weight less than 1 lies between the tips from \( v_1 \) and \( v_5 \). Any move involving a vertex \( v \) with weight less than 1 yields a vertex with weight less than 1 at \( v \) or a neighbor of \( v \). Thus, after any subsequent move involving the vertex with weight less than 1, the tips from \( v_1 \) and \( v_5 \) are still separated by a vertex with weight less than 1. It follows that the tips from \( v_1 \) and \( v_5 \) cannot reach the same vertex, so five tips cannot reach the same vertex. Thus \( a_f(P) \geq \lceil n/4 \rceil \), and consequently \( a_f(P) = \lceil n/4 \rceil \).

A similar proof works for cycles.

**Theorem 9.** If \( G \) is the \( n \)-vertex cycle, then \( a_f(G) = \lceil n/4 \rceil \).
Proof. In [1], Lampert and Slater showed that $a_t(C_n) = \lceil n/4 \rceil$, so we know that $a_f(C_n) \leq \lceil n/4 \rceil$.

Suppose that there are two tips $t_1$ and $t_2$ with heights $h_1$ and $h_2$, respectively, and let $h_1 \geq h_2$. Any fractional acquisition move involving a vertex $v$ with weight less than $h_2$ leaves either $v$ or a neighbor of $v$ with weight less than $h_2$. Therefore the existence of a cutset with weights less than $h_2$ in $C_n$ is preserved by fractional acquisition moves. Thus, if there is a cutset in $C_n$ consisting of vertices with weights less than $h_2$ that separates $t_1$ and $t_2$, then the tips cannot reach the same vertex.

Let $v_1, \ldots, v_5$ be five vertices in $C_n$, labeled cyclically. Let $v_i^{+j}$ and $v_i^{-j}$ denote the vertices on $C$ that are distance $j$ from $v_i$ in the direction of $v_{i+1}$ and $v_{i-1}$, respectively (with addition performed modulo 5). For $i \in [5]$, let $t_i$ be the tip from $v_i$, and let $h_i$ be the height of $t_i$. Consider a fractional protocol $A$ on $C$. If $A$ produces a cutset of vertices with weight less than 1 between two of the five tips in question, then the tips cannot reach the same vertex. Thus, when the first tip among $\{t_1, \ldots, t_5\}$ moves, we may assume that there is not a cutset of vertices with weight less than 1 separating any two of the tips.

By symmetry, we may assume that $t_1$ is the first of the five tips to move, and that $t_1$ moves to $v_1^{+1}$. Now $h_1 > 1$ and there is a vertex (namely $v_1$) with weight less than 1 that lies on one of the two paths joining the vertices holding $t_1$ and $t_5$. If there is a vertex with weight less than 1 on the path $v_i^{+2}v_i^{+3} \ldots v_i^{-1}$, then the tips cannot reach the same vertex. Thus, all of the vertices in $\{v_i^{+2}, v_i^{+3}, \ldots, v_5\}$ have weight at least 1. Furthermore, since any move involving the vertices in $\{v_i^{+2}, v_i^{+3}, \ldots, v_5\}$ generates a vertex with weight less than 1, there have been no such moves, and all of the vertices in $\{v_i^{+2}, v_i^{+3}, \ldots, v_5\}$ have weight exactly 1 and $w(v_5) \geq 1$. At this point, any move that transfers weight from one of the vertices in $\{v_i^{+2}, v_i^{+3}, \ldots, v_5\}$ produces a cutset of vertices with weights less than $\min\{h_1, h_5\}$ separating the vertices holding $h_1$ and $h_5$. It follows that the tips $t_3, t_4,$ and $t_5$ cannot move without guaranteeing a cutset of vertices with low weight between $t_1$ and $t_5$. Thus the five tips $t_1$ and $t_5$ cannot reach the same vertex, and consequently no five tips can reach the same vertex. Therefore $a_f(C_n) = \lceil n/4 \rceil$. \hfill \qed

We now show that the proofs of Theorems 8 and 9, unlike in the case of total acquisition, cannot be achieved by bounding the amount of weight that a single vertex can acquire, as is the case for the total acquisition number of paths and cycles. For this proof we again use the normalized model.

**Theorem 10.** There is a fractional protocol on $P_{4t+2}$ that yields a vertex with weight at least $t$.

**Proof.** We will prove that there is a fractional protocol that achieves a distribution with an ascending path of length $t + 3$ where $t + 2$ of the vertices have positive weight. We begin by defining a family of normalized weight distributions. Label the vertices of $P_{4k+2}$ in order as $v_1, \ldots, v_{4k+2}$. The normalized weight distribution $w$ on $P_{4k+2}$ is a wave if the following conditions hold.

1. $w(v_2) − w(v_1) = 2$;
2. \( w(v_{2i}) = w(v_{2i+1}) \) for all \( i \in \{1, \ldots, 2t\} \);

3. \( w(v_{2i}) > 0 \) if \( i \) is odd and \( w(v_{2i}) < 0 \) if \( i \) is even.

We show that if there is a wave weight distribution \( w \) on \( P_{4k+2} \), then there is a normalized protocol \( \mathcal{W}_{4k+2} \) called the wave protocol that does not change the weight on \( v_1 \) and yields a wave weight distribution on the \( 4(k - 1) + 2 \)-vertex subpath on the vertices \( \{v_2, \ldots, v_{4k-1}\} \). An example of the wave protocol on \( P_{10} \) is illustrated in Figure 4. We describe \( \mathcal{W}_{4k+2} \); it consists of \( 2k - 1 \) steps performed in order, each step consisting of several normalized acquisition moves. Let \( w' \) be the weight distribution generated by \( \mathcal{W}_{4k+2} \). For step 1, move weight 1 from \( v_2 \) to \( v_3 \), and set

\[
\begin{align*}
    w'(v_1) &= w(v_1); \\
    w'(v_2) &= w(v_2) - 1; \\
    w'(v_3) &= w(v_3) + 1.
\end{align*}
\]

For the remaining \( 2k - 2 \) steps in \( \mathcal{W}_{4k+2} \), we perform the following operations in order:

1. If \( i \) is even, then transfer weight \( w'(v_{2i-1}) - w(v_{2i}) \) from \( v_{2i+1} \) to \( v_{2i} \) and set

\[
\begin{align*}
    w'(v_{2i}) &= w'(v_{2i-1}); \\
    w'(v_{2i+1}) &= w(v_{2i+1}) - (w'(v_{2i-1}) - w(v_{2i})).
\end{align*}
\]

2. If \( i \) is odd, then transfer weight \( w(v_{2i}) - w'(v_{2i-1}) \) from \( v_{2i} \) to \( v_{2i+1} \) and set

\[
\begin{align*}
    w'(v_{2i}) &= w'(v_{2i-1}); \\
    w'(v_{2i+1}) &= w(v_{2i+1}) + (w(v_{2i}) - w'(v_{2i-1})).
\end{align*}
\]

Figure 4: The wave protocol on \( P_{10} \). The relative weight of each vertex represented by its vertical position. The final figure is the resulting wave distribution on \( \{v_2, \ldots, v_7\} \).

We now show that \( w' \) is a wave on the \( 4(k - 1) + 2 \)-vertex subpath consisting of the vertices \( \{v_2, \ldots, v_{4k-1}\} \). First, since \( w(v_2) = w(v_3) \), weight 1 moves from \( v_2 \) to \( v_3 \), and no other moves
involve \(v_2\) or \(v_3\), we observe that \(w'(v_3) - w'(v_2) = 2\). Second, by the design of \(A\), it is clear that \(w'(v_{2i+1}) = w'(v_{2i+2})\) for all \(i \in \{1, \ldots, 2k - 2\}\). Finally, if \(i\) is odd, then \(w'(v_{2i+1}) = w(v_{2i+1}) + (w(v_{2i}) - w'(v_{2i-1}))\). By assumption, \(w(v_{2i+1}) = w(v_{2i})\) and \(w(v_{2i}) > 0\). Therefore \(w'(v_{2i+1}) > 0\). If \(i\) is even, then \(w'(v_{2i+1}) = w(v_{2i+1}) - (w'(v_{2i-1}) - w(v_{2i}))\). By assumption, \(w(v_{2i+1}) = w(v_{2i})\) and \(w(v_{2i}) < 0\). Therefore \(w(v_{2i+1}) < 0\).

To obtain a vertex in \(P_{4t+2}\) with weight at least \(t + 2\), we describe a normalized protocol \(A\) that yields a normalized weight distribution that contains an ascending path containing at least \(t + 2\) vertices with nonnegative weight. In step 0 of \(A\), we initialize by moving weight 1 from \(v_i\) to \(v_{i+1}\) when \(i \equiv 1 \pmod{4}\) and weight 1 from \(v_i\) to \(v_{i-1}\) when \(i \equiv 0 \pmod{4}\). This yields a wave weight distribution on \(P_{4t+1}\).

For each \(i \in \{1, \ldots, t - 1\}\), in step \(i\) perform the protocol \(W_{4(t-i+1)+2}\) on the subpath on the vertices \(\{v_i, \ldots, v_{4t+2-3(i-1)}\}\). As shown above, after each iteration, the next subpath has a wave weight distribution. Furthermore, since \(v_2\) has weight 0 after step 1, it follows that the vertices in \(\{v_2, \ldots, v_{t+1}\}\) all have nonnegative weight. Furthermore, \(w(v_1) < w(v_2) < \ldots < w(v_{t+1})\). By Lemma 3, there is a fractional protocol on \(P_{4t+4}\) that yields a strictly ascending path on \(\{v_1, \ldots, v_{t+1}\}\) in which the vertices in \(\{v_2, \ldots, v_{t+1}\}\) all have weight at least 1. By Lemma 2, \(v_{t+1}\) can acquire all of the weight on this path, so \(v_{t+1}\) can acquire weight at least \(t\).

4 Open Questions and Conjectures

Though we have determined the fractional acquisition number of all graphs, there are many open questions to consider. In particular, no consideration has been given to the length of the protocols in the proofs in this paper. The following extremal question naturally arises.

**Question 1.** If \(G\) is a connected graph with maximum degree at least 3, what is the minimum number of fractional acquisition moves in a fractional protocol that moves all of the weight in \(G\) to a single vertex?

It is reasonable to allow fractional acquisition moves on edges that are far apart to happen simultaneously. More precisely, a set of fractional acquisition moves can occur in parallel if the edges used by the moves all have distinct endpoints. Within this framework, Question 1 can be recast in terms of run time.

**Question 2.** Assume that every fractional acquisition move takes a single unit of time, and moves may be performed in parallel. What is the minimum run time of a fractional protocol that transfers all of the weight to a single vertex in a given graph \(G^*\)?

On the other hand, one can bound the number of fractional acquisition moves allowed.
Question 3. Let $G$ be an $n$-vertex graph and let $f : \mathbb{N} \rightarrow \mathbb{N}$ be a function. What is the minimum number of vertices in $G$ with positive weight after a fractional protocol of $f(n)$ fractional acquisition moves? What is the maximum amount of weight that a vertex in $G$ can acquire using at most $f(n)$ acquisition moves?

There is a fractional protocol on the $n$-vertex star that uses $n - 1$ moves and transfers all of the weight to a single vertex. Furthermore, moving all of the weight in an $n$-vertex graph to a single vertex requires at least $n - 1$ fractional acquisition moves. Among trees it is natural to consider the graphs that require the maximum number of fractional acquisition moves. This is similar in spirit to the LeSaulnier-West characterization of the $n$-vertex trees that maximize the total acquisition number [3].

Question 4. Which $n$-vertex trees with maximum degree at least 3 require the maximum number of fractional acquisition moves to transfer all of the weight to a single vertex?

We note that the proof of Theorem 10 is not optimized to move the maximum amount of weight to a single vertex. Indeed, more carefully described versions of the protocol in Theorem 10 do consolidate more weight on a single vertex. However, the analysis of such protocols is far more complicated than the proof of Theorem 10 and yields the same result, namely that one cannot bound the amount of weight that a vertex in a graph with maximum degree 2 can acquire. For the maximum amount of weight that a vertex in a path can acquire, we do make the following conjecture.

Conjecture 11. For $n \geq 3$, the amount of weight that any vertex in $P_n$ can acquire via fractional acquisition moves is bounded by $\left\lfloor \frac{n}{2} \right\rfloor + 2$, and this bound is best possible.

References


