Saturation numbers in tripartite graphs

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Abstract

Given graphs $H$ and $F$, a subgraph $G \subseteq H$ is an $F$-saturated subgraph of $H$ if $F \not\subseteq G$, but $F \subseteq G + e$ for all $e \in E(H) \setminus E(G)$. The saturation number of $F$ in $H$, denoted $\text{sat}(H, F)$, is the minimum number of edges in an $F$-saturated subgraph of $H$. In this paper we study saturation numbers of tripartite graphs in tripartite graphs. For $\ell \geq 1$ and $n_1$, $n_2$, and $n_3$ sufficiently large, we determine $\text{sat}(K_{n_1,n_2,n_3}, K_{\ell,\ell,\ell})$ and $\text{sat}(K_{n_1,n_2,n_3}, K_{\ell,\ell,\ell-1})$ exactly and $\text{sat}(K_{n_1,n_2,n_3}, K_{\ell,\ell,\ell-2})$ within an additive constant. We also include general constructions of $K_{\ell,m,p}$-saturated subgraphs of $K_{n_1,n_2,n_3}$ with few edges for $\ell \geq m \geq p > 0$.

Keywords: 05C35; saturation; tripartite; subgraph

1 Introduction

In this paper, all graphs are simple and we let $V(G)$ and $E(G)$ denote the vertex set and edge set of the graph $G$, respectively. Let $\overline{G}$ denote the complement of $G$. For a set of vertices $S \subseteq V(G)$, we let $G[S]$ denote the induced subgraph of $G$ on $S$.

Given a graph $F$, a graph $G$ is $F$-saturated if $F$ is not a subgraph of $G$ but $F$ is a subgraph of $G + e$ for any edge $e \in E(\overline{G})$. The saturation number of $F$ is the minimum size of an $n$-vertex $F$-saturated graph, and is denoted $\text{sat}(n, F)$. Saturation numbers were first studied by Erdős, Hajnal, and Moon [3], who proved that $\text{sat}(n, K_k) = (k-2)n - {k-1 \choose 2}$ and characterized the $n$-vertex $K_k$-saturated graphs with this number of edges. For a thorough account of the results known about saturation numbers, the reader should consult the excellent survey of Faudree, Faudree, and Schmitt [4].

Because saturation numbers consider the addition of any edge from $\overline{G}$ to $G$, it is natural in this setting to think of $G$ as a subgraph of the complete graph $K_n$. In this paper we consider saturation numbers when $G$ is treated as a subgraph of a complete tripartite graph.

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Let $F$ and $H$ be graphs be fixed graphs; we call $H$ the host graph. A subgraph $G$ of $H$ is an $F$-saturated subgraph of $H$ if $F$ is not a subgraph of $G$, but $F$ is a subgraph of $G + e$ for all $e \in E(H) \setminus E(G)$. The saturation number of $F$ in $H$ is the minimum number of edges in an $F$-saturated subgraph of of $H$, and is denoted $\text{sat}(H, F)$. With this notation, $\text{sat}(n, F) = \text{sat}(K_n, F)$.

The first result on saturation numbers in host graphs that are not complete is from a related problem in bipartite graphs. Let $\text{sat}(K_{(n_1, n_2)}, K_{(\ell, m)})$ denote the minimum number of edges in a bipartite $G$ graph on the vertex set $V_1 \cup V_2$ where $|V_i| = n_i$ such that: 1) $G$ does not contain $K_{\ell,m}$ with $\ell$ vertices in $V_1$ and $m$ vertices in $V_2$, and 2) the addition of any edge joining $V_1$ and $V_2$ yields a copy of $K_{\ell,m}$ with $\ell$ vertices in $V_1$ and $m$ vertices in $V_2$. This parameter is the minimization analogue of the Zarankiewicz number. Bollobás and Wessel [1, 2, 8, 9] independently proved that $\text{sat}(K_{(n_1, n_2)}, K_{(\ell, m)}) = (m - 1)n_1 + (\ell - 1)n_2 - (m - 1)(\ell - 1)$ for $2 \leq \ell \leq n_1$ and $2 \leq m \leq n_2$, confirming a conjecture of Erdős, Hajnal, and Moon from [3].

In [5], Moshkovitz and Shapira studied saturation numbers in $d$-uniform $d$-partite hypergraphs. When $d = 2$, this reduces to saturation numbers of bipartite graphs in bipartite graphs. They provided a construction showing that $\text{sat}(K_{n,n}, K_{\ell,m}) \leq (\ell + m - 2)n - \left[ \left( \frac{\ell + m - 2}{2} \right)^2 \right]$ and conjectured that the bound is sharp for $n$ sufficiently large. This upper bound shows that for $n$ sufficiently large, $\text{sat}(K_{n,n}, K_{\ell,m}) < \text{sat}(K_{(n,n)}, K_{(\ell,m)})$. Recently, Gan, Korándi and Sudakov [6] showed that $\text{sat}(K_{n,n}, K_{\ell,m}) \geq (\ell + m - 2)n - (\ell + m - 2)^2$ and proved that the Moshkovitz-Shapira bound is sharp for $K_{2,3}$, the first nontrivial case.

Let $K^n_k$ denote the complete $k$-partite graph in which each partite set has order $n$. In [5], Ferrara, Jacobson, Pfender, and the second author studied the saturation number of $K_3$ in balanced multipartite graphs. They proved that if $k \geq 3$ and $n \geq 100$, then

$$\text{sat}(K^n_k, K_3) = \min\{2kn + n^2 - 4k - 1, 3kn - 3n - 6\}.$$ 

Furthermore, they characterized the $K_3$-saturated subgraphs of $K^n_k$ of minimum size.

The focus of this paper is the saturation numbers in complete tripartite graphs. In Section 2, we provide constructions of $K_{\ell,m,p}$-saturated subgraphs of $K_{n_1, n_2, n_3}$ with small size. In Section 3, we determine $\text{sat}(K_{n_1, n_2, n_3}, K_{\ell, \ell, \ell})$ and $\text{sat}(K_{n_1, n_2, n_3}, K_{\ell, \ell, \ell-1})$ and characterize the $K_{\ell,\ell,\ell}$-saturated subgraphs and $K_{\ell,\ell,\ell-1}$-saturated subgraphs of $K_{n_1, n_2, n_3}$ of minimum size. In Section 4, we prove that for $\text{sat}(K_{n,n,n}, K_{\ell,\ell,\ell-2})$, the upper bound obtained from the construction in Section 2 is correct within an additive constant depending on $\ell$. Finally, Section 5 contains various conjectures and open questions for future work.

Throughout the paper, we will assume that $n_1 \geq n_2 \geq n_3$, and that the partite sets of
$K_{n_1,n_2,n_3}$ are $V_1$, $V_2$, and $V_3$ with $|V_i| = n_i$. We label the vertices in $V_i$ as $V_i = \{v_i^1, \ldots, v_i^{n_i}\}$. When $G$ is a tripartite graph on the vertex set $V_1 \cup V_2 \cup V_3$, we let $\delta_i(G)$ denote the minimum degree of the vertices in $V_i$. When the graph in question is clear, we simply write $\delta_i$. For a vertex $v \in G$, we let $N_i(v)$ denote the set of neighbors of $v$ in set $V_i$; that is, $N_i(v) = N(v) \cap V_i$. Similarly, if $S$ is a set of vertices in $G$, then $N_i(S) = \bigcup_{v \in S} N_i(v)$. Throughout the paper, all arithmetic in subscripts is performed modulo 3. We also use $[k]$ to denote the set $\{1, \ldots, k\}$.

2 Constructions of saturated subgraphs of $K_{n_1,n_2,n_3}$

This section contains constructions of $K_{\ell,m,p}$-saturated subgraphs of $K_{n_1,n_2,n_3}$ with few edges. We begin with two constructions of $K_{\ell,m,p}$-saturated subgraphs of $K_{n_1,n_2,n_3}$ when $m = p$. The reader is invited to keep in mind the particular case of $K_{\ell,\ell,\ell}$, in which the constructions are greatly simplified and which we prove are best possible in Section 3.

Construction 1. Let $\ell$ and $m$ be positive integers such that $\ell \geq m$. Let $n_1 \geq n_2 \geq n_3 \geq \max\{\ell + 2, 3\ell - 2m - 2\}$. For each $i \in [3]$, let $S_i$ be the $m$-vertex set $\{v_i^{n_i-m+1}, \ldots, v_i^{n_i}\}$ and join $S_i$ to $V_{i+1}$ and $V_{i+2}$. When $\ell > m$, add the following edges, where arithmetic in the superscripts of vertices in $V_i$ is performed modulo $n_i - m$:

1. for $a \in [n_3 - m]$, join $v_3^a$ to $\{v_1^a, \ldots, v_1^{a+\ell-m-1}\} \cup \{v_2^a, \ldots, v_2^{a+\ell-m-1}\}$;
2. for $a \in [n_2 - m]$, join $v_2^a$ to $\{v_1^{a+\ell-m}, \ldots, v_1^{a+2\ell-2m-1}\}$.

Finally, in all cases, remove the edges $v_1^{n_1}v_2^{n_2}$, $v_1^{n_1}v_3^{n_3}$, and $v_2^{n_2}v_3^{n_3}$ (see Figure 1). We call this graph $G_1$.

For a set of integers $S$, let $S \mod n$ denote the set of residues of the elements of $S$ modulo $n$. Thus we have

$$E(G_1) = \{v_i^j v_j^r : i \in [3], j \in [3], i \ne j, n_i - m + 1 \leq r \leq n_i \text{ or } n_j - m + 1 \leq s \leq n_j\}$$

$$\cup \{v_3^a v_j^r : j \in \{1,2\}, a \in [n_3 - m], b \in \{a, \ldots, a + \ell - m - 1\} \mod (n_j - m)\}$$

$$\cup \{v_2^a v_j^r : a \in [n_2 - m], b \in \{a + \ell - m, \ldots, a + 2\ell - 2m - 1\} \mod (n_1 - m)\}$$

$$\setminus \{v_1^{n_1}v_2^{n_2}, v_1^{n_1}v_3^{n_3}, v_1^{n_1}v_3^{n_3}, v_2^{n_2}v_3^{n_3}\}.$$
Figure 1: Construction 1: A $K_{\ell,m,m}$-saturated subgraph of $K_{n_1,n_2,n_3}$. Solid lines denote complete joins between sets, and dotted lines denote edges that have been removed. The lines marked with “max degree $\ell - m$” represent the edges described in items 1 and 2 of Construction 1.

Construction 2. For $i \in [3]$, let $G_i^j$ be the graph obtained from the graph from Construction 1 by removing the set $\{v_i^{n_1}, v_i^{n_1+1}, v_i^{n_i-1}, v_i^{n_i+1}, v_i^{n_i+2}, v_i^{n_i+2}\}$ instead of $\{v_1^{n_1}, v_2^{n_2}, v_1^{n_1}, v_2^{n_2}, v_3^{n_3}\}$ (see Figure 2).

Theorem 1. Let $\ell$ and $m$ be positive integers such that $\ell \geq m$. For $n_1 \geq n_2 \geq n_3 \geq \max\{\ell + 2, 3\ell - 2m - 1\}$, the graphs from Construction 1 and Construction 2 are $K_{\ell,m,m}$-saturated subgraphs of $K_{n_1,n_2,n_3}$. Thus,

$$\text{sat}(K_{n_1,n_2,n_3}, K_{\ell,m,m}) \leq 2m(n_1 + n_2 + n_3) + (\ell - m)(n_2 + 2n_3) - 3\ell m - 3.$$ 

Proof. Let $G$ be a graph from Construction 1 or 2. By construction, $G - (S_1 \cup S_2 \cup S_3)$ is triangle-free. Therefore, if $v \in V_i \setminus S_i$, then $G[N(v)]$ does not contain $K_{\ell,m}$ as a subgraph. Since $G[S_i \cup S_{i+1}]$ is not a complete bipartite graph, it then follows that $G$ is $K_{\ell,m,m}$-free.

Let $e = uv$ be a nonedge in $G$. We show that $G + e$ contains $K_{\ell,m,m}$; there are two cases to consider.

Case 1: $e$ joins two vertices in $S_1 \cup S_2 \cup S_3$. If $e$ joins $S_i$ and $S_{i+1}$, then $G + e$ contains $K_{\ell,m,m}$ on the vertices $\{v_i^1, \ldots, v_i^{\ell}\} \cup S_i \cup S_{i+1}$.

Case 2: $e$ joins two vertices in $V(G) \setminus (S_1 \cup S_2 \cup S_3)$. Let $i, j \in [3]$ such that $i < j$, and assume that $e = v_j^a v_i^b$ where $a \in [n_j - m]$ and $b \in [n_i - m]$. Let $k$ be the third value in $[3]$. 

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Figure 2: Construction 2: A $K_{\ell,m,m}$-saturated subgraph of $K_{n_1,n_2,n_3}$. Solid lines denote complete joins between sets, and dotted lines denote edges that have been removed. The lines marked with “max degree $\ell - m$” represent the edges described in items 1 and 2 of Construction 1.

Let $x_i \in S_i$ and $x_j \in S_j$ be the vertices that have a nonneighbor in $S_k$. By construction, $S_i - x_i$ is completely joined to $S_j - x_j$. In this case, $G + e$ contains $K_{\ell,m,m}$ on the vertex set $(N_i(v^a_i) + v^b_i - x_i) \cup (S_j + v^a_j - x_j) \cup S_k$.

We now construct $K_{\ell,m,p}$-saturated subgraphs of $K_{n_1,n_2,n_3}$ when $m > p$. Like Constructions 1 and 2, the subgraph of this construction induced by $(V_1 \setminus S_1) \cup (V_2 \setminus S_2) \cup (V_3 \setminus S_3)$ consists of bipartite graphs with maximum degree $\ell - m$. Unlike Constructions 1 and 2, the vertices in this set have fewer than $\ell$ neighbors in the other partite sets. Therefore it is not necessary to specify completely the neighborhoods of these vertices.

**Construction 3.** Let $\ell$, $m$, and $p$ be positive integers such that $\ell \geq m > p$. Let $n_1 \geq n_2 \geq n_3 \geq \ell$. For each $i \in [3]$ let $S_i$ be an $(m - 1)$-vertex subset of $V_i$ and join $S_i$ to $V_{i+1}$ and $V_{i+2}$. For $i < j$, join $V_i \setminus S_i$ to $V_j \setminus S_j$ with an $(\ell - m)(n_j - m + 1)$-edge graph with maximum degree $\ell - m$. Thus each vertex in $V_j \setminus S_j$ has exactly $\ell - m$ neighbors in $V_i \setminus S_i$, and each vertex in $V_i \setminus S_i$ has at most $\ell - m$ neighbors in $V_j \setminus S_j$.

**Theorem 2.** Let $\ell$, $m$, and $p$ be positive integers such that $\ell \geq m > p$. For $n_1 \geq n_2 \geq n_3 \geq \ell$, the graph from Construction 3 is a $K_{\ell,m,p}$-saturated subgraph of $K_{n_1,n_2,n_3}$. Thus,

$$\text{sat}(K_{n_1,n_2,n_3}, K_{\ell,m,p}) \leq 2(m - 1)(n_1 + n_2 + n_3) + (\ell - m)(n_2 + 2n_3) - 3\ell(m - 1) + 3m - 3.$$
Figure 3: Construction 3: A $K_{\ell,m,p}$-saturated subgraph of $K_{n_1,n_2,n_3}$ for $m > p$. Solid lines denote complete joins between sets. The lines marked with “max degree $\ell - m$” represent the $(\ell - m)(n_j - m + 1)$-edge graphs with maximum degree $\ell - m$ used in Construction 3.

\[ \text{Proof.} \] Let $G$ be the graph described in Construction 3. Let $i \in [3]$. If $v \in V_i \setminus S_i$, then $v$ has at most $\ell - 1$ neighbors in $V_{i+1}$ and at most $\ell - 1$ neighbors in $V_{i+2}$. Since there are only $m - 1$ vertices in $S_i$, it follows that $G$ does not contain $K_{\ell,m}$, and therefore $G$ is $K_{\ell,m,p}$-free.

Let $i, j \in [3]$ such that $i < j$, and let $k$ be the third value in $[3]$. Let $e$ be a nonedge in $G$ joining $v_i \in V_i$ and $v_j \in V_j$. Thus $G + e$ contains $K_{\ell,m,m-1}$ on the vertex set $(N_i(v_j) + v_i) \cup (S_j + v_j) \cup S_k$. Since $m > p$, it follows that $G + e$ contains $K_{\ell,m,p}$. \qed

We include two final constructions in the special case of $K_{\ell,m,p}$-saturated subgraphs of $K_{n,n,n}$. These constructions are inspired by the $K_{\ell,m}$-saturated subgraphs of $K_{n,n}$ used in [7] and [6]. When the host graph is balanced, Constructions 1, 2, and 3 contain large $(\ell - m)$-regular graphs; we will replace those graphs with graphs with slightly fewer edges.

**Construction 4.** Let $\ell$ and $m$ be positive integers such that $\ell \geq m$ and let

$$n \geq \max \left\{ \ell + 2, 3\ell + \left\lfloor \frac{\ell - m}{2} \right\rfloor - 2m - 2 \right\}.$$

For each $i \in [3]$, let $S_i = \{v_i^1, \ldots, v_i^m\}$ and join $S_i$ to $V_{i+1}$ and $V_{i+2}$. Let $t = \left\lfloor \frac{\ell - m}{2} \right\rfloor$, and for each $i \in [3]$ let $T_i = \{v_i^{m+1}, \ldots, v_i^{m+t}\}$. For all $i \in [3]$, completely join $T_i$ to $T_{i+1}$. Let $\bigcup_{i \in [3]}(V_i \setminus (S_i \cup T_i))$ span a triangle-free tripartite graph so that for all $i \in [3]$, each vertex in $V_i \setminus (S_i \cup T_i)$ has exactly $\ell - m$ neighbors in both $V_{i+1} \setminus (S_{i+1} \cup T_{i+1})$ and $V_{i+2} \setminus (S_{i+2} \cup T_{i+2})$. 

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(such a graph is easily obtained using items 1 and 2 from Construction 1). Finally, remove the edges \{v_1^1v_2^1, v_1^1v_3^1, v_2^1v_3^1\} (see Figure 4).

Figure 4: Construction 4: A $K_{\ell,m,m}$-saturated subgraph of $K_{n,n,n}$. Solid lines denote complete joins between sets, and dotted lines denote edges that have been removed. The lines marked with "$(\ell - m)$-regular" represent the triangle-free tripartite graph used in Construction 4.

It is possible to modify Construction 4 so that the edges removed induce $P_4$ rather than $K_3$ as in Construction 2 (for instance, remove \{v_1^iv_{i+1}^1, v_2^iv_{i+1}^2, v_{i+1}^iv_{i+2}^1\}). Since we do not prove that these constructions are best possible nor that they characterize the $K_{\ell,m,m}$-saturated subgraphs of $K_{n,n,n}$ of minimum size, we do not include this variant as a separate construction.

We now present a $K_{\ell,m,p}$-saturated subgraph of $K_{n,n,n}$ for $m > p$.

**Construction 5.** Let $\ell$, $m$, and $p$ be positive integers such that $\ell \geq m > p$ and let $n \geq \ell + \left\lfloor \frac{\ell-m}{2} \right\rfloor - 1$. For each $j \in [3]$, let $S_j$ be an $(m-1)$-vertex subset of $V_j$ and join $S_i$ to $V_{i+1}$ and $V_{i+2}$. Let $t = \left\lfloor \frac{\ell-m}{2} \right\rfloor$, and for each $j \in [3]$ let $T_i$ be a $t$-vertex subset of $V_j \setminus S_j$. For all $i \in [3]$, completely join $T_i$ to $T_{i+1}$. For each $i \in [3]$, let $(V_i \cup V_{i+1}) \setminus (S_i \cup S_{i+1} \cup T_i \cup T_{i+1})$ induce an $(\ell - m)$-regular bipartite graph.

Constructions 4 and 5 yield the following two theorems. The proofs of these theorems follow almost immediately from the proofs of Theorems 1 and 2, respectively, and therefore we omit them.
Theorem 3. Let \( \ell \) and \( m \) be positive integers such that \( \ell \geq m \) and let
\[
n \geq \max \left\{ \ell + 2, 3\ell + \left\lfloor \frac{\ell - m}{2} \right\rfloor - 2m - 2 \right\}.
\]
The graph from Construction 4 is a \( K_{\ell,m,m} \)-saturated subgraph of \( K_{n,n,n} \), and thus
\[
sat(K_{n,n,n}, K_{\ell,m,p}) \leq 3(\ell + m)n - 3\left( \ell - m - \left\lfloor \frac{\ell - m}{2} \right\rfloor \right) \left\lfloor \frac{\ell - m}{2} \right\rfloor - 3\ell m - 3.
\]

Theorem 4. Let \( \ell, m, \) and \( p \) be positive integers such that \( \ell \geq m > p \) and let \( n \geq \ell + \left\lfloor \frac{\ell - m}{2} \right\rfloor - 1 \). The graph from Construction 5 is a \( K_{\ell,m,p} \)-saturated subgraph of \( K_{n,n,n} \), and thus
\[
sat(K_{n,n,n}, K_{\ell,m,p}) \leq 3(\ell + m - 2)n - 3(m - 1)(\ell - 1) + 3 \left\lfloor \frac{\ell - m}{2} \right\rfloor^2 - 3(\ell - m) \left\lfloor \frac{\ell - m}{2} \right\rfloor.
\]

Figure 5: Construction 5: A \( K_{\ell,m,p} \)-saturated subgraph of \( K_{n,n,n} \). Solid lines denote complete joins between sets. The lines marked with “\((\ell - m)\)-regular” represent the \((\ell - m)\)-regular bipartite graphs used in Construction 5.

3 The saturation numbers of \( K_{\ell,\ell,\ell} \) and \( K_{\ell,\ell,\ell-1} \)

In this section we prove the following two theorems on saturation numbers in tripartite graphs.
Theorem 5. Let $\ell$ be a positive integer. If $n_1$, $n_2$, and $n_3$ are positive integers such that $n_1 \geq n_2 \geq n_3 \geq 32\ell^2 + 40\ell^2 + 11\ell$, then

$$\text{sat}(K_{n_1,n_2,n_3}, K_{\ell,\ell,\ell}) = 2\ell(n_1 + n_2 + n_3) - 3\ell^2 - 3.$$ 

Furthermore, the graphs from Constructions 1 and 2 are the only $K_{\ell,\ell,\ell}$-saturated subgraphs of $K_{n_1,n_2,n_3}$ with this number of edges.

Theorem 6. Let $\ell$ be a positive integer. If $n_1$, $n_2$, and $n_3$ are positive integers such that $n_1 \geq n_2 \geq n_3 \geq 32(\ell - 1)^3 + 40(\ell - 1)^2 + 11(\ell - 1)$, then

$$\text{sat}(K_{n_1,n_2,n_3}, K_{\ell,\ell,\ell-1}) = 2(\ell - 1)(n_1 + n_2 + n_3) - 3(\ell - 1)^2.$$ 

Furthermore, the graph from Construction 3 is the unique $K_{\ell,\ell,\ell-1}$-saturated subgraph of $K_{n_1,n_2,n_3}$ with this number of edges.

Though $K_{\ell,\ell,\ell}$ and $K_{\ell,\ell,\ell-1}$ correspond to different constructions from Section 2, they are both of the form $K_{\ell,\ell,m}$ for $\ell \geq m$. Thus we begin by establishing some common lemmas on the number of edges in $K_{\ell,\ell,m}$-saturated subgraphs of $K_{n_1,n_2,n_3}$ when $m \geq 1$.

Lemma 7. Let $i \in [3]$ and assume that $n_i \geq (3m + 1)(\delta_{i+1} + \delta_{i+2}) + 2m^2 + m$. If $G$ is a $K_{\ell,\ell,m}$-saturated subgraph of $K_{n_1,n_2,n_3}$ such that $\delta_i > 2m$, then $|E(G)| \geq 2m(n_1 + n_2 + n_3)$.

Proof. For each $j \in [3]$, let $v_j$ be a vertex of degree $\delta_j$ in $V_j$. Each nonneighbor of $v_i$ in $V_{i+1} \cup V_{i+2}$ must have at least $m$ common neighbors with $v_i$. Therefore there are at least $m(n_{i+1} + n_{i+2} - \delta_i)$ edges joining $V_{i+1}$ and $V_{i+2}$. Similarly, there are at least $m(n_{i+1} - \delta_{i+2})$ edges joining $V_{i+1}$ and $N_i(v_{i+2})$ and at least $m(n_{i+2} - \delta_{i+1})$ edges joining $V_{i+2}$ and $N_i(v_{i+1})$. Finally, there are at least $\delta_i(n_i - \delta_{i+1} - \delta_{i+2})$ edges incident to $V_i \setminus (N_i(v_{i+1}) \cup N_i(v_{i+2}))$. Summing, we have

$$|E(G)| \geq m(2n_{i+1} + 2n_{i+2} - \delta_{i+1} - \delta_{i+2}) + \delta_i(n_i - \delta_{i+1} - \delta_{i+2} - m).$$

Since $n_i > \delta_{i+1} + \delta_{i+2} + m$, this lower bound is increasing in $\delta_i$. Therefore, if $\delta_i > 2m$, then

$$|E(G)| \geq m(2n_{i+1} + 2n_{i+2} - \delta_{i+1} - \delta_{i+2}) + (2m + 1)(n_i - \delta_{i+1} - \delta_{i+2} - m)$$

$$\geq 2m(n_1 + n_2 + n_3) + n_i - [(3m + 1)(\delta_{i+1} + \delta_{i+2}) + 2m^2 + m]$$

$$\geq 2m(n_1 + n_2 + n_3).$$

Lemma 8. Let $n_1 \geq n_2 \geq n_3 \geq 32m^3 + 40m^2 + 11m$. If $G$ is a $K_{\ell,\ell,m}$-saturated subgraph of $K_{n_1,n_2,n_3}$ such that $\delta_i > 2m$ for some $i \in [3]$, then $|E(G)| \geq 2m(n_1 + n_2 + n_3)$.
Proof. First observe that each vertex in $V_i$ has at least $m$ neighbors in both $V_{i+1}$ and $V_{i+2}$ or is completely joined to $V_{i+1}$ or $V_{i+2}$. Thus $\delta(G) \geq 2m$. There are two cases to consider depending on the order of $n_i$.

**Case 1:** $n_1 \leq 4mn_2$. If $\delta_1 \geq 6m$, then $|E(G)| \geq 6mn_1 \geq 2m(n_1 + n_2 + n_3)$. Therefore we may assume that $\delta_1 < 6m$. If $\delta_2 \geq 8m^2 + 4m$, then $|E(G)| \geq (8m^2 + 4m)n_2 \geq 2m(n_1 + n_2 + n_3)$. Therefore we may assume that $\delta_2 < 8m^2 + 4m$. Since $n_3 \geq (3m + 1)(8m^2 + 10m) + 2m^2 + m$, Lemma 7 implies that if $\delta_3 > 2m$, then $|E(G)| \geq 2m(n_1 + n_2 + n_3)$. Therefore we may assume that $\delta_3 = 2m$. Lemma 7 now implies that if $\delta_1 > 2m$ or $\delta_2 > 2m$, then $|E(G)| \geq 2m(n_1 + n_2 + n_3)$.

**Case 2:** $n_1 > 4mn_2$. If $\delta_1 > 2m$, then $|E(G)| \geq (2m + 1)n_1 \geq 2m(n_1 + n_2 + n_3)$. Therefore we may assume that $\delta_1 = 2m$. Let $R$ be the set of vertices in $V_1$ with degree $2m$. If $|V_1 \setminus R| \geq 2m(n_2 + n_3)$, then $|E(G)| \geq 2m(n_1 + n_2 + n_3)$. Therefore we assume that $|V_1 \setminus R| < 2m(n_2 + n_3)$.

If $v \in R$, then each vertex in $N_2(v)$ is adjacent to every vertex in $V_3 \setminus N_3(v)$. Thus each vertex in $N_2(R)$ has at least $n_3 - m$ neighbors in $V_3$. If $|N_2(R)| \geq 4mn_2/(n_3 - m)$, then there are at least $4mn_2$ edges joining $V_2$ and $V_3$, and consequently $|E(G)| \geq 2m(n_1 + n_2 + n_3)$. Therefore we may assume that $|N_2(R)| < 4mn_2/(n_3 - m)$.

There are at least $\delta_2(n_2 - 4mn_2/(n_3 - m))$ edges incident to $V_2 \setminus N_2(R)$. There are at least $2m(n_1 - 2m(n_2 + n_3))$ edges incident to $R$. Therefore, if $\delta_2 \geq 8m^2 + 4m + 1$, then

\[
|E(G)| \geq 2m(n_1 - 2m(n_2 + n_3)) + (8m^2 + 4m + 1) \left( n_2 - \frac{4mn_2}{n_3 - m} \right)
\geq 2mn_1 + 4mn_2 + n_2 - n_2 \left( \frac{8m^2 + 4m + 1}{n_3 - m} \right)
\geq 2m(n_1 + n_2 + n_3).
\]

Therefore we may assume that $\delta_2 \leq 8m^2 + 4m$.

Since $\delta_1 = 2m$, $\delta_2 \leq 8m^2 + 4m$, and $n_3 \geq (3m + 1)(8m^2 + 6m) + 2m^2 + m$, Lemma 7 implies that if $\delta_3 > 2m$, then $|E(G)| \geq 2m(n_1 + n_2 + n_3)$. Therefore we may assume that $\delta_3 = 2m$. It now follows from Lemma 7 that if $\delta_2 > 2m$, then $|E(G)| \geq 2m(n_1 + n_2 + n_3)$. 

We now prove Theorems 5 and 6.

**Proof of Theorem 5.** Let $G$ be a $K_{\ell, \ell, \ell}$-saturated subgraph of $K_{n_1, n_2, n_3}$ of minimum size. It follows from Lemma 8 that if $\delta_i > 2\ell$ for any $i \in [3]$, then $|E(G)| \geq 2\ell(n_1 + n_2 + n_3)$. Since it is clear that $\delta(G) \geq 2\ell$, we assume that $\delta_1 = \delta_2 = \delta_3 = 2\ell$. 

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For $i \in [3]$, let $v_i \in V_i$ be a vertex of degree $2\ell$. Thus $v_i$ has $\ell$ neighbors in $V_{i+1}$ and $\ell$ neighbors in $V_{i+2}$, and $G$ contains all edges joining $N_{i+1}(v_i)$ to $V_{i+2} \setminus N_{i+2}(v_i)$ and all edges joining $N_{i+2}(v_i)$ to $V_{i+1} \setminus N_{i+1}(v_i)$. Therefore, the vertices of degree $2\ell$ in $G$ form an independent set. Let $S = N(v_1) \cup N(v_2) \cup N(v_3)$ and let $S_i = S \cap V_i$. Since $v_{i+1}$ and $v_{i+2}$ have $\ell$ common neighbors, we conclude that $N_i(v_{i+1}) = N_i(v_{i+2})$ and therefore $|S_i| = \ell$. Since the addition of an edge joining $v_i$ and any vertex in $(V_{i+1} \cup V_{i+2}) \setminus N(v_i)$ completes a copy of $K_{\ell,\ell,\ell}$, there are at least $\ell^2 - 1$ edges joining $S_{i+1}$ and $S_{i+2}$. Therefore there are at least $\ell(n_{i+1} + n_{i+2}) - \ell^2 - 1$ edges joining $V_{i+1}$ and $V_{i+2}$. Thus $|E(G)| \geq 2\ell(n_1 + n_2 + n_3) - 3\ell^2 - 3$, and in conjunction with Theorem 1 we conclude that $\text{sat}(K_{n_1,n_2,n_3}, K_{\ell,\ell,\ell}) = 2\ell(n_1 + n_2 + n_3) - 3\ell^2 - 3$. 

Since $|E(G)| = 2\ell(n_1 + n_2 + n_3) - 3\ell^2 - 3$, it follows that there are exactly $\ell^2 - 1$ edges joining $S_i$ and $S_{i+1}$ for all $i \in [3]$. Suppose that $G$ is not isomorphic to a graph from Construction 1 or 2. Thus the three nonedges in $G[S]$ do not induce $K_3$ or $P_4$. Without loss of generality, assume that $u_1^1u_1^1$ is a nonedge in $G[S]$ and the other two nonedges in $G[S]$ are incident to $u_1^2$ and $u_1^3$, respectively. Since $G$ is $K_{\ell,\ell,\ell}$-saturated, there is a subgraph $H$ of $G + v_i v_{i+1}$ that is isomorphic to $K_{\ell,\ell,\ell}$. It follows that $H$ must contain $v_i$, $v_{i+1}$ and $S_{i+2}$, and therefore $H$ cannot contain $u_i^2$ or $u_i^3$. Since $H$ must contain $\ell$ neighbors of $v_i$ in $V_{i+1}$ and $u_i^2 \notin H$, we conclude that $u_{i+1}^1 \in H$. Similarly, it follows that $u_i^1 \in H$. However, this implies that $H$ contains the nonedge $u_i^1 u_i^1$, a contradiction. Therefore, $G$ is isomorphic to a graph from Construction 1 or 2.

**Proof of Theorem 6.** Let $G$ be a $K_{\ell,\ell,\ell-1}$-saturated subgraph of $K_{n_1,n_2,n_3}$ of minimum size. It follows from Lemma 8 that if $\delta_i > 2(\ell - 1)$ for any $i \in [3]$, then $|E(G)| \geq 2(\ell - 1)(n_1 + n_2 + n_3)$. It is clear that $\delta(G) \geq 2(\ell - 1)$, and thus we assume that $\delta_1 = \delta_2 = \delta_3 = 2(\ell - 1)$.

For $i \in [3]$, let $v_i \in V_i$ be a vertex of degree $2(\ell - 1)$. Thus $v_i$ has $\ell - 1$ neighbors in $V_{i+1}$ and $\ell - 1$ neighbors in $V_{i+2}$, and $G$ contains all edges joining $N_{i+1}(v_i)$ to $V_{i+2} \setminus N_{i+2}(v_i)$ and all edges joining $N_{i+2}(v_i)$ to $V_{i+1} \setminus N_{i+1}(v_i)$. Therefore, the vertices of degree $2(\ell - 1)$ in $G$ form an independent set. Let $S = N(v_1) \cup N(v_2) \cup N(v_3)$ and let $S_i = S \cap V_i$. Since $v_{i+1}$ and $v_{i+2}$ have $\ell - 1$ common neighbors, we conclude that $N_i(v_{i+1}) \cup N_i(v_{i+2})$ and therefore $|S_i| = \ell - 1$. Furthermore, since the addition of an edge joining $v_i$ and a vertex in $V_{i+1} \setminus N_{i+1}(v_i)$ yields a copy of $K_{\ell,\ell,\ell-1}$, it follows that $N_{i+1}(v_i)$ and $N_{i+2}(v_i)$ must be completely joined. Thus, $S_i$ and $S_{i+1}$ are completely joined for all $i \in [3]$. Therefore the graph from Construction 4 is a subgraph of $G$. Since $G$ is $K_{\ell,\ell,\ell-1}$-saturated, it follows that $G$ is isomorphic to the graph from Construction 4, and therefore $\text{sat}(K_{n_1,n_2,n_3}, K_{\ell,\ell,\ell-1}) = 2(\ell - 1)(n_1 + n_2 + n_3) - 3(\ell - 1)^2$.

We note that it is possible to lower the bounds on $n_3$ in Theorems 5 and 6 through a
more careful analysis of the algebra in Lemmas 7 and 8. However, this appears still to yield a lower bound on \( n_3 \) that is cubic in \( \ell \), and mainly distracts from the main ideas of the proof.

## 4 The saturation number of \( K_{\ell,\ell,\ell-2} \)

In this section we prove that the graph from Construction 5 is within an additive constant of the minimum number of edges in a \( K_{\ell,\ell,\ell-2} \)-saturated subgraph of \( K_{n,n,n} \). Given two sets of vertices \( S \) and \( T \), we let \([S,T]\) denote the set of edges with one endpoint in \( S \) and one endpoint in \( T \).

**Theorem 9.** Let \( \ell \) be a positive integer. For \( n \) sufficiently large,

\[
\text{sat}(K_{n,n,n}, K_{\ell,\ell,\ell-2}) \geq 6(\ell - 1)n - (72\ell^2 - 40\ell + 54).
\]

**Proof.** Let \( G \) be a \( K_{\ell,\ell,\ell-2} \)-saturated subgraph of \( K_{n,n,n} \). If \( \delta_i(G) \geq 6(\ell - 1) \) for some \( i \in [3] \), then \( |E(G)| \geq 6(\ell - 1)n \). Therefore we may assume that \( \delta_i < 6(\ell - 1) \) for all \( i \in [3] \), and consequently a vertex of degree \( \delta_i \) in \( V_i \) must have nonneighbors in both \( V_{i+1} \) and \( V_{i+2} \). Assume that \( v \) is a vertex of degree at most \( 2\ell - 3 \) in \( V_i \). If \( |N_{i+1}(v)| < \ell - 2 \), the addition of an edge joining \( v \) and \( V_{i+2} \) does not complete a copy of \( K_{\ell,\ell,\ell-2} \). Therefore we may assume without loss of generality that \( 2\ell - 4 \leq d(v) \leq 2\ell - 3 \) and \( v \) has \( \ell - 2 \) neighbors in \( V_{i+1} \) and at most \( \ell - 1 \) neighbors in \( V_{i+2} \). It follows that the addition of an edge joining \( v \) and \( V_{i+1} \) does not complete a copy of \( K_{\ell,\ell,\ell-2} \), and therefore \( G \) is not \( K_{\ell,\ell,\ell-2} \)-saturated. We conclude that \( \delta_i \geq 2\ell - 2 \) for all \( i \in [3] \).

Let \( c = 72\ell^2 - 40\ell + 54 \). If \( |[V_i, V_{i+1}]| \geq 2(\ell - 1)n - c/3 \) for all \( i \in [3] \), then \( |E(G)| \geq 6(\ell - 1)n - c \). Therefore we may assume that \( |[V_{i+1}, V_{i+2}]| < 2(\ell - 1)n - c/3 \) for some \( i \in [3] \). Let \( v \in V_i \) have degree \( \delta_i \). Every vertex in \( V_{i+1} \setminus N_{i+1}(v_i) \) is adjacent to at least \( \ell - 2 \) vertices in \( N_{i+2}(v_i) \). If \( v' \) is a vertex in \( V_i \) that has only \( \ell - 2 \) neighbors in \( V_{i+2} \), then each vertex in \( V_{i+2} \setminus N_{i+2}(v') \) has \( \ell \) neighbors in \( N_{i+1}(v') \). Therefore

\[
|[V_{i+1}, V_{i+2}]| \geq (\ell - 2)(n - \delta_i) + \ell(n - \delta_i - \ell + 2) \\
\geq 2(\ell - 1)n - ((2\ell - 2)\delta_i + \ell^2 - 2\ell) \\
\geq 2(\ell - 1)n - (13\ell^2 - 26\ell + 12),
\]

a contradiction. Therefore we assume that every vertex in \( V_i \) has at least \( \ell - 1 \) neighbors in \( V_{i+2} \), and by symmetry, also in \( V_{i+1} \).
Let \( X_i^0 = N(v_i) \). For \( k \geq 1 \), recursively define \( X_i^k \) to be the vertices in \( (V_{i+1} \cup V_{i+2}) \setminus (\bigcup_{j=0}^{k-1} X_i^j) \) that have at least \( \ell - 1 \) neighbors in \( \bigcup_{j=0}^{k-1} X_i^j \). Define \( X_i \) to be the set of vertices that are in \( X_i^k \) for any value of \( k \). By definition, \( G[X_i] \) contains at least \( (\ell - 1)(|X_i| - \delta_i) \) edges.

Let \( R_i = (V_{i+1} \cup V_{i+2}) \setminus X_i \). Note that each vertex in \( R_i \) is adjacent to exactly \( \ell - 2 \) vertices in \( N(v_i) \). Let \( T_{i,1}, \ldots, T_{i,a_i} \) be the components of \( G[R_i] \) that are trees. Thus \( G[R_i] \) contains at least \( |R_i| - a_i \) edges, and

\[
|V_{i+1}, V_{i+2}| \geq (\ell - 1)(2n - \delta_i) - a_i \geq 2(\ell - 1)n - 6(\ell - 1)^2 - a_i. \tag{1}
\]

If \( T_{i,b} \) consists of a single vertex \( v \in V_{i+1} \) and \( T_{i,b'} \) consists of a single vertex \( u \in V_{i+2} \), then the addition of \( uv \) cannot complete a copy of \( K_{\ell,\ell-2} \) in \( G \). Therefore, since \( N_{i+1}(v_i) \) and \( N_{i+2}(v_i) \) are nonempty,

\[
a_i \leq \max\{|R_i \cap V_{i+1}|, |R_i \cap V_{i+2}|\} < n. \tag{2}
\]

Observe that

\[
|E(G)| \geq \sum_{j=1}^{a_i} (|E(T_{i,j})| + ||V(T_{i,j}), V_i)||).
\]

If \( |E(T_{i,j})| + ||V(T_{i,j}), V_i)| > 6(\ell - 1)n/a_i \) for all \( j \in [a_i] \), then \( |E(G)| > 6(\ell - 1)n \). Therefore we assume that there is a component \( T_{i,k_i} \) of \( G[R_i] \) such that \( |E(T_{i,k_i})| + |E(T_{i,k_i}, V_i)| \leq 6(\ell - 1)n/a_i \). Thus \( |V(T_{i,k_i})| \leq 6(\ell - 1)n/a_i + 1 \). If \( x \in V_{i+2} \cap V(T_{i,k_i}) \) and \( w \in V_{i+1} \setminus V(T_{i,k_i}) \), then the addition of \( xw \) cannot complete a copy of \( K_{\ell,\ell-2} \) in \( V_{i+1} \cup V_{i+2} \). Therefore each vertex in \( w \in V_{i+1} \setminus V(T_{i,k_i}) \) has at least \( \ell \) neighbors in \( N_i(x) \). Observe that \( |N_i(x)| \leq 6(\ell - 1)n/a_i \).

Similarly, for \( x \in V_{i+1} \cap V(T_{i,k_i}) \), every vertex in \( V_{i+2} \setminus V(T_{i,k_i}) \) has at least \( \ell \) neighbors in \( N_i(x) \), and \( |N_i(x)| \leq 6(\ell - 1)n/a_i \). We consider two cases.

**Case 1:** For some \( i \in 3 \), \( |V_{i+1}, V_{i+2}| < 2(\ell - 1)n - c/3 \) and \( T_{i,k_i} \) contains vertices in both \( V_{i+1} \) and \( V_{i+2} \). Let \( x_{i+1} \in V_{i+1} \cap V(T_{i,k_i}) \) and let \( x_{i+2} \in V_{i+2} \cap V(T_{i,k_i}) \). Therefore

\[
\sum_{v \in V_i} d(v) \geq \delta_i(n - d_i(x_{i+1}) - d_i(x_{i+2})) + \ell(n - d_{i+2}(x_{i+1})) + \ell(n - d_{i+1}(x_{i+2}))
\]

\[
\geq 2(\ell - 1)(n - 12(\ell - 1)n/a_i) + 2\ell(n - 6(\ell - 1)n/a_i)
\]

Summing the edges we have

\[
|E(G)| \geq |V_{i+1}, V_{i+2}| + \sum_{v \in V_i} d(v)
\]

\[
\geq 2(\ell - 1)n - 6(\ell - 1)^2 - a_i + 2(\ell - 1)(n - 12(\ell - 1)n/a_i) + 2\ell(n - 6(\ell - 1)n/a_i)
\]

\[
\geq -a_i + (6(\ell - 1) + 2)n - 6(\ell - 1)^2 - (36\ell^2 - 60\ell + 24)n/a_i.
\]

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If \(|E(G)| < 6(\ell - 1)n\), then we conclude that

\[ a_i < (n - 3(\ell - 1)^2) - \sqrt{(n - 3(\ell - 1)^2)^2 - (36\ell^2 - 60\ell + 24)n} \quad \text{or} \quad a_i > (n - 3(\ell - 1)^2) + \sqrt{(n - 3(\ell - 1)^2)^2 - (36\ell^2 - 60\ell + 24)n}. \]

From (2) we know that \(a_i < n\), so we conclude that for \(n\) sufficiently large,

\[ a_i < (n - 3(\ell - 1)^2) - \sqrt{(n - 3(\ell - 1)^2)^2 - (36\ell^2 - 60\ell + 24)n}. \]

Since

\[ \lim_{n \to \infty} (n - 3(\ell - 1)^2) - \sqrt{(n - 3(\ell - 1)^2)^2 - (36\ell^2 - 60\ell + 24)n} = 18\ell^2 - 30\ell + 12, \]

it follows from the integrality of \(a_i\) that for \(n\) sufficiently large, \(a_i \leq 18\ell^2 - 30\ell + 12\). Therefore \(|[V_{i+1}, V_{i+2}]| \geq 2(\ell - 1)n - 6(\ell - 1)^2 - (18\ell^2 - 30\ell + 12) \geq 2(\ell - 1)n - c/3\), a contradiction.

**Case 2:** For some \(i \in 3\), \(|[V_{i+1}, V_{i+2}]| < 2(\ell - 1)n - c/3\) and \(T_{i,k_i} \cap V_{i+1} = \emptyset\) or \(T_{i,k_i} \cap V_{i+2} = \emptyset\). Without loss of generality we assume that \(|[V_2, V_3]| < 2(\ell - 1)n - c/3\) and \(T_{1,k_1} \cap V_3 = \emptyset\). Thus \(T_{1,k_1}\) consists of a single vertex in \(V_2\) that has only \(\ell - 2\) neighbors in \(V_3\); call this vertex \(x\). Furthermore, \(d(x) \leq 6(\ell - 1)n/a_1\). Since the addition of an edge joining \(x\) to \(V_3\) cannot complete a copy of \(K_{\ell,\ell}\) in \(V_2 \cup V_3\), each nonneighbor of \(x\) in \(V_3\) has at least \(\ell\) neighbors in \(N_1(x)\). Since every vertex in \(V_1\) has at least \(\ell - 1\) neighbors in \(V_3\), we conclude that \(|[V_1, V_3]| \geq (2(\ell - 1)n - 6(\ell - 1)n/a_1)\). Consequently,

\[ |E(G)| = |[V_1, V_2]| + |[V_1, V_3]| + |[V_2, V_3]| \]

\[ \geq |[V_1, V_2]| + (2(\ell - 1)n - 6(\ell - 1)n/a_1) + (2(\ell - 1)n - 6(\ell - 1)^2 - a_1) \]

\[ = |[V_1, V_2]| + 4(\ell - 1)n + n - (12\ell^2 - 18\ell + 6)n/a_1 - 6(\ell - 1)^2 - a_1. \]

First assume that \(|[V_1, V_2]| \geq 2(\ell - 1)n - c/3\). If \(|E(G)| < 6(\ell - 1)n - c\), then

\[ 0 \geq -a_1 + n - 6(\ell - 1)^2 + 2c/3 - (12\ell^2 - 18\ell + 6)n/a_1, \]

which requires

\[ a_1 < \frac{1}{2} \left(n - 6(\ell - 1)^2 + 2c/3 - \sqrt{(n - 6(\ell - 1)^2 + 2c/3)^2 - (48\ell^2 - 72\ell + 24)n}\right) \quad \text{or} \quad (3) \]

\[ a_1 > \frac{1}{2} \left(n - 6(\ell - 1)^2 + 2c/3 + \sqrt{(n - 6(\ell - 1)^2 + 2c/3)^2 - (48\ell^2 - 72\ell + 24)n}\right). \quad (4) \]

Since \(c \geq 45\ell^2 - 72\ell + 27\), it follows that \(2c/3 \geq 30\ell^2 - 48\ell + 18 \geq 24\ell^2 - 36\ell + 12 + 6(\ell - 1)^2\). Therefore, if inequality (4) holds, then \(a_1 \geq n\). This violates inequality (2), so we conclude that

\[ a_1 < \frac{1}{2} \left(n - 6(\ell - 1)^2 + 2c/3 - \sqrt{(n - 6(\ell - 1)^2 + 2c/3)^2 - (48\ell^2 - 72\ell + 24)n}\right). \]
Since
\[
\lim_{n \to \infty} \frac{n - 6(\ell - 1)^2 + \frac{2}{3}c - \sqrt{(n - 6(\ell - 1)^2 + \frac{2}{3}c)^2 - (48\ell^2 - 72\ell + 24)n}}{2} = 12\ell^2 - 18\ell + 6,
\]
it follows from the integrality of \(a_1\) that for \(n\) sufficiently large, \(a_1 \leq 12\ell^2 - 18\ell + 6\). Therefore \(|[V_2, V_3]| \geq 2(\ell - 1)n - 6(\ell - 1)^2 - (12\ell^2 - 18\ell + 6) \geq 2(\ell - 1)n - c/3\), a contradiction.

Now assume that \(|[V_1, V_2]| < 2(\ell - 1)n - c/3\). Therefore \(T_{3,k_3}\) exists. If \(T_{3,k_3}\) contains vertices in both \(V_1\) and \(V_2\), then by Case 1 we conclude that \(|E(G)| \geq 6(\ell - 1)n - c\). Therefore we assume that \(T_{3,k_3}\) contains a single vertex \(y \in V_1 \cup V_2\), and \(d(y) \leq 6(\ell - 1)n/a_3\). Since every vertex in \(V_1\) has at least \(\ell - 1\) neighbors in both \(V_2\) and \(V_3\) and \(y\) has only \(\ell - 2\) neighbors in \(V_1 \cup V_2\), we conclude that \(y \in V_2\).

The \(n - (\ell - 2)\) nonneighbors of \(x\) in \(V_3\) each have at least \(\ell\) neighbors in \(N_1(x)\). Similarly, each vertex in \(V_1 \setminus (N_1(x) \cup N_1(y))\) has at least \(\ell\) neighbors in \(N_3(y)\). Since \(|V_1 \setminus (N_1(x) \cup N_1(y))| \geq n - 6(\ell - 1)n/a_1 - (\ell - 2)\), we conclude that
\[
|[V_1, V_3]| \geq 2\ell n - 6\ell(\ell - 1)n/a_1 - 2\ell(\ell - 2).
\]
Using inequalities (1) and (2), we have
\[
|E(G)| = |[V_1, V_3]| + |[V_2, V_3]| + |[V_1, V_2]|
\]
\[
\geq (2\ell n - 6\ell(\ell - 1)n/a_1 - 2\ell(\ell - 2)) + (4(\ell - 1)n - 12(\ell - 1)^2 - a_1 - a_3)
\]
\[
\geq -a_1 + 6(\ell - 1)n + 2n - a_3 - 14\ell^2 - 28\ell + 12 - 6\ell(\ell - 1)n/a_1
\]
\[
\geq -a_1 + 6(\ell - 1)n + n - (14\ell^2 - 28\ell + 12) - 6\ell(\ell - 1)n/a_1.
\]
Therefore \(|E(G)| < 6(\ell - 1)n - c\) only if
\[
a_1 < \frac{1}{2} \left( n + c - (14\ell^2 - 28\ell + 12) - \sqrt{(n + c - (14\ell^2 - 28\ell + 12))^2 - 24\ell(\ell - 1)n} \right)
\]
(5)
\[
a_1 > \frac{1}{2} \left( n + c - (14\ell^2 - 28\ell + 12) + \sqrt{(n + c - (14\ell^2 - 28\ell + 12))^2 - 24\ell(\ell - 1)n} \right).
\]
(6)
Since \(c \geq 26\ell^2 - 40\ell + 12\), it follows that \(c - (14\ell^2 - 28\ell + 12) \geq 12\ell(\ell - 1)\). Therefore, if inequality (6) holds, then \(a_1 \geq n\). This violates inequality (2), so we conclude that
\[
a_1 < \frac{1}{2} \left( n + c - (14\ell^2 - 28\ell + 12) - \sqrt{(n + c - (14\ell^2 - 28\ell + 12))^2 - 24\ell(\ell - 1)n} \right).
\]
Since
\[
\lim_{n \to \infty} \frac{n + c - (14\ell^2 - 28\ell + 12) - \sqrt{(n + c - (14\ell^2 - 28\ell + 12))^2 - 24\ell(\ell - 1)n}}{2} = 6\ell(\ell-1),
\]
it follows from the integrality of \(a_1\) that for \(n\) sufficiently large, \(a_1 \leq 6\ell(\ell - 1)\). Therefore \(|V_2, V_3| \geq 2(\ell - 1)n - 6(\ell - 1)^2 - 6\ell(\ell - 1) \geq 2(\ell - 1)n - c/3\), a contradiction.
5 Conclusion

We conclude with several open questions and conjectures. First, we conjecture that in a sufficiently large, sufficiently unbalanced host graph, the constructions in Section 2 are best possible.

Conjecture 10. Let \( \ell \) and \( m \) be positive integers such that \( \ell > m \). For \( n_1 \geq n_2 \geq n_3 \) sufficiently large compared to \( \ell \), and \( n_1 \) sufficiently large compared to \( n_3 \),

\[
\text{sat}(K_{n_1,n_2,n_3}, K_{\ell,m,m}) = 2m(n_1 + n_2 + n_3) + (\ell - m)(n_2 + 2n_3) - 3\ell m - 3.
\]

Conjecture 11. Let \( \ell, m, \) and \( p \) be positive integers such that \( \ell \geq m > p \). For \( n_1 \geq n_2 \geq n_3 \), \( n_3 \) sufficiently large compared to \( \ell \), and \( n_1 \) sufficiently large compared to \( n_3 \),

\[
\text{sat}(K_{n_1,n_2,n_3}, K_{\ell,m,p}) = 2(m - 1)(n_1 + n_2 + n_3) + (\ell - m)(n_2 + 2n_3) - 3\ell(m - 1) + 3m - 3.
\]

Following the direction taken in [5], one can study the saturation number of \( K_{\ell,m,p} \) in \( k \)-partite graphs for \( k > 3 \). The following is the logical place to begin such research.

Question 1. Let \( K_k^n \) denote the complete \( k \)-partite graph in which all partite sets have size \( n \). For \( \ell \geq 2, k \geq 4, \) and \( n \) sufficiently large, what is \( \text{sat}(K_k^n, K_{\ell,\ell,\ell}) \)?

We also note that if \( G \) is a graph with chromatic number at most 3, then determining \( \text{sat}(K_{n_1,n_2,n_3}, G) \) is nontrivial. Thus it is natural to consider the saturation number of bipartite graphs in complete tripartite graphs. As a first example, we compute the saturation number of \( C_4 \) in tripartite graphs.

Proposition 12. For \( n_1 \geq n_2 \geq n_3 \geq 2 \),

\[
\text{sat}(K_{n_1,n_2,n_3}, C_4) = n_1 + n_2 + n_3.
\]

Proof. It is clear that a \( C_4 \)-saturated subgraph of \( K_{n_1,n_2,n_3} \) must be connected, and no spanning tree of \( K_{n_1,n_2,n_3} \) is \( C_4 \)-saturated. It is also straightforward to check that the graph with edge set \( \{v_i^1v_{i+1}^3 | i \in [3], j \in [n_{i+1}] \} \) is \( C_4 \)-saturated (see Figure 6).

Observe that \( \text{sat}(K_{n_1,n_2,n_3}, C_4) \) and the sharpness example are not obtained using the bipartite saturation number of \( C_4 \). Thus it appears that the study of saturation numbers of bipartite graphs in tripartite graphs will differ from the work initiated in [6] and [7].
Figure 6: A $C_4$-saturated subgraph of $K_{n_1,n_2,n_3}$. Solid lines denote complete joins between two sets.

References


[9] W. Wessel, Über eine Klasse paarer Graphen. II. Bestimmung der Minimalgraphen, 