Colored Saturation Parameters for Rainbow Subgraphs

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December 30, 2014

Abstract

Inspired by a 1987 result of Hanson and Toft [Edge-colored saturated graphs, \textit{J. Graph Theory} \textbf{11} (1987), 191–196] and several recent results, we consider the following saturation problem for edge-colored graphs. An edge-coloring of a graph \( F \) is \textit{rainbow} if every edge of \( F \) receives a different color. Let \( \mathcal{R}(F) \) denote the set of rainbow-colored copies of \( F \). A \( t \)-edge-colored graph \( G \) is \( (\mathcal{R}(F), t) \)-\textit{saturated} if \( G \) does not contain a rainbow copy of \( F \) but for any edge \( e \in E(G) \) and any color \( i \in [t] \), the addition of \( e \) to \( G \) in color \( i \) creates a rainbow copy of \( F \). Let \( \text{sat}_t(n, \mathcal{R}(F)) \) denote the minimum number of edges in an \( (\mathcal{R}(F), t) \)-saturated graph of order \( n \). We call this the \textit{rainbow saturation number} of \( F \).

In this paper, we prove several results about rainbow saturation numbers of graphs. In stark contrast with the related problem for monochromatic subgraphs, wherein the saturation is always linear in \( n \), we prove that rainbow saturation numbers have a variety of different orders of growth. For instance, the rainbow saturation number of the complete graph \( K_n \) lies between \( n \log n/\log \log n \) and \( n \log n \), the rainbow saturation number of an \( n \)-vertex star is quadratic in \( n \), and the rainbow saturation number of any tree that is not a star is linear.

\textbf{Keywords:} saturation; edge-coloring; rainbow

1 Introduction

All graphs considered in this paper are simple. For a positive integer \( t \), we let \( [t] \) denote the set \( \{1, \ldots, t\} \). The degree of a vertex \( v \) will be denoted \( d(v) \), and the minimum and maximum degree of a graph \( G \) will be denoted \( \delta(G) \) and \( \Delta(G) \), respectively. A \textit{t-edge-coloring} of a graph

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G is a function $f : E(G) \to [t]$, and a graph equipped with such a coloring is a $t$-edge-colored graph. In this paper, we do not require edge-colorings to be proper edge-colorings.

Given a family of graphs $\mathcal{F}$, a graph $G$ is $\mathcal{F}$-saturated if no $F \in \mathcal{F}$ is a subgraph of $G$, but for any $e \in E(G)$, some $F \in \mathcal{F}$ is a subgraph of $G + e$. The minimum number of edges in an $n$-vertex $\mathcal{F}$-saturated graph is the saturation number of $\mathcal{F}$ and is denoted $\text{sat}(n, \mathcal{F})$. If $\mathcal{F} = \{F\}$, then we instead say that $G$ is $F$-saturated, and write $\text{sat}(n, F)$. Saturation numbers were introduced by Erdős, Hajnal, and Moon in [3], where they determined the saturation number of $K_k$ and characterized the $n$-vertex $K_k$-saturated graphs of this size. Since then, saturation numbers have received considerable attention; for more results we refer the reader to the dynamic survey of Faudree, Faudree, and Schmitt [5].

In [7], Hanson and Toft extended the notion of saturation numbers to edge-colored graphs. Given a family $\mathcal{C}$ of edge-colored graphs, we say that a $t$-edge-colored graph $G$ is $(\mathcal{C}, t)$-saturated if $G$ contains no member of $\mathcal{C}$ as a (colored) subgraph, but for any edge $e \in E(G)$ and any color $i \in [t]$, the addition of $e$ to $G$ in color $i$ creates some member of $\mathcal{C}$. Let $\text{sat}_t(n, \mathcal{C})$ denote the minimum number of edges in a $(\mathcal{C}, t)$-saturated graph of order $n$. We call this the $t$-edge-colored saturation number of $\mathcal{C}$. Following [6], we will refer to a coloring of $G$ with this property as a $\mathcal{C}$-threshold coloring.

Let $\mathcal{M}(H_1, \ldots, H_t)$ denote the set of edge-colored graphs consisting of one copy each of $H_1, \ldots, H_t$ such that all edges in $H_i$ are colored with color $i$ for all $i \in [t]$. In [7], Hanson and Toft proved the following theorem for the saturation number of monochromatic complete graphs.

**Theorem 1** (Hanson and Toft [7]). Let $t$ be a positive integer. If $k_i \geq 2$ is a positive integer for $1 \leq i \leq t$ and $k = \sum k_i$, then

$$\text{sat}_t(n, \mathcal{M}(K_{k_1}, \ldots, K_{k_t})) = \begin{cases} \binom{n}{2} & \text{if } n \leq k - 2t \\ \binom{k-2t}{2} + (k - 2t)(n - k + 2t) & \text{if } n > k - 2t. \end{cases}$$

It is important to recall that $(\mathcal{C}, t)$-saturated graphs are already edge-colored. However, for Theorem 1, the condition that the addition of any edge in any color yields a monochromatic complete graph bears a striking resemblance to the main ideas of Ramsey theory, in which the graphs do not have a specified coloring.

Given graphs $G$ and $H_1, \ldots, H_t$, we write $G \rightarrow (H_1, \ldots, H_t)$ if every $t$-edge coloring of $G$ contains a copy of $H_i$ that is monochromatic in color $i$ for some $i \in [t]$. Thus the Ramsey number $r(k_1, \ldots, k_t)$ is the minimum $n$ such that $K_n \rightarrow (K_{k_1}, \ldots, K_{k_t})$. A graph $G$ is $(H_1, \ldots, H_t)$-Ramsey-minimal if $G \rightarrow (H_1, \ldots, H_t)$ but $G - e \not\rightarrow (H_1, \ldots, H_t)$ for all $e \in E(G)$. The set of $(H_1, \ldots, H_t)$-Ramsey-minimal graphs is denoted $\mathcal{R}_{\text{min}}(H_1, \ldots, H_t)$. 2
Hanson and Toft made the following conjecture.

**Conjecture 1.** If \( r = r(k_1, \ldots, k_t) \) is the standard Ramsey number for complete graphs, then

\[
\text{sat}(n, R_{\text{min}}(K_{k_1}, \ldots, K_{k_t})) = \begin{cases} 
\binom{n}{2} & \text{if } n < r \\
(r-2) + (r-2)(n-r+2) & \text{if } n \geq r.
\end{cases}
\]

A graph that is \( R_{\text{min}}(H_1, \ldots, H_t) \)-saturated, which is not edge-colored, has an edge coloring that does not contain a copy of \( H_i \) for any \( i \in [t] \). However, the addition of any edge \( e \) to \( G \) yields a subgraph \( G' \) such that \( G' \rightarrow (H_1, \ldots, H_t) \), so that every \( t \)-edge-coloring of \( G + e \) must contain a monochromatic copy of \( H_i \) in color \( i \) for some \( i \in [t] \).

In [2], Chen et al. verified the Hanson–Toft conjecture for \( \text{sat}(n, R_{\text{min}}(K_3, K_3)) \), the first nontrivial case. They also proved an upper bound on \( \text{sat}(n, R_{\text{min}}(K_{t}, T_m)) \) where \( T \) is a tree of order \( m \) and determined \( \text{sat}(n, R_{\text{min}}(K_3, P_3)) \). More recently [6], Ferrara, Kim and Yeager determined \( \text{sat}(n, R_{\text{min}}(m_1K_2, \ldots, m_kK_2)) \) for \( m_1, \ldots, m_k \geq 1 \) and \( n > 3(m_1+\ldots+m_k-k) \), and characterized the saturated graphs of minimum size.

In this paper we consider saturation numbers of edge-colored graphs that are as far from being monochromatic as possible. An edge-coloring of a graph \( F \) is rainbow if every edge of \( F \) receives a different color. Let \( \mathcal{R}(F) \) denote the set of rainbow-colored copies of \( F \); as a technical detail, note that it is not necessary to specify the set of colors that may be used to edge-color \( F \) in \( \mathcal{R}(F) \). In this paper we study \( \text{sat}_t(n, \mathcal{R}(F)) \), the \( t \)-edge-colored saturation number for rainbow copies of \( F \). Informally we refer to this as the rainbow saturation number of \( F \).

Observe that for \( F \neq K_2 \), a monochromatic complete graph does not contain an element of \( \mathcal{R}(F) \) and therefore is vacuously \( (\mathcal{R}(F), t) \)-saturated. Also, the empty graph is \( (\mathcal{R}(K_2), t) \)-saturated. Therefore \( \text{sat}_t(n, \mathcal{R}(F)) \) is defined whenever \( F \) is nonempty.

Note that every noncomplete \( R_{\text{min}}(H_1, \ldots, H_t) \)-saturated graph necessarily has an \( M(H_1, \ldots, H_t) \)-threshold coloring. Consequently,

\[
\text{sat}(n, M(H_1, \ldots, H_t)) \leq \text{sat}(n, R_{\text{min}}(H_1, \ldots, H_t)).
\]

In [8], Kásonyi and Tuza proved that for any nonempty family of graphs \( \mathcal{F} \), \( \text{sat}(n, \mathcal{F}) = O(n) \), which therefore implies that

\[
\text{sat}(n, M(H_1, \ldots, H_t)) = O(n).
\]

Our results show that this is unequivocally not the case for rainbow saturation numbers. In Section 2 we study the rainbow saturation number of complete graphs, proving that their
order of growth lies between \( n \log n / \log \log n \) and \( n \log n \). In Section 4 we establish linear upper bounds on the rainbow saturation number of various families of graphs, including trees with at least five vertices that are not stars. In Section 3, we consider the rainbow saturation number of some additional graphs, including stars, whose rainbow saturation numbers are quadratic in \( n \), and matchings, whose rainbow saturation numbers do not depend on \( n \). Section 5 contains several open questions and conjectures.

2 Complete Graphs

In this section we study the rainbow saturation number of complete graphs. The following theorem is the main result of this section.

**Theorem 2.** Let \( k \) be a positive integer that is at least 3, and let \( t \) be an integer that is at least \( \binom{k}{2} \). For all \( n \) sufficiently large, there exist constants \( c_1 \) and \( c_2 \) such that

\[
c_1 \frac{n \log n}{\log \log n} \leq \text{sat}_t(n, \mathcal{R}(K_k)) \leq c_2 n \log n.
\]

Theorem 2 follows from immediately from Theorems 3 and 4 below, each of which regards a broader class of graphs than cliques alone. Theorem 3, which applies to all graphs in which every edge lies on a triangle, supplies the lower bound. The upper bound follows from Theorem 4, which applies to all connected graphs that have no independent vertex cut.

**Theorem 3.** Let \( F \) be a graph with the property that every edge in \( F \) lies on a triangle. For every \( t \geq |E(F)| \) there exists a constant \( c = c(F) \) such that

\[
\text{sat}_t(n, \mathcal{R}(F)) \geq cn \frac{\log n}{\log \log n}.
\]

**Proof.** Suppose that \( G \) is an \( (\mathcal{R}(F), t) \)-saturated graph of order \( n \). We wish to show that \( G \) has \( \Omega \left( n \frac{\log n}{\log \log n} \right) \) edges. This is a nearly immediate consequence of the following claim.

Claim 1. Let \( G \) be an edge-colored graph such that for every pair of nonadjacent vertices \( x \) and \( y \) in \( V(G) \), there are two internally disjoint rainbow paths of length 2 joining \( x \) and \( y \) whose edges have four distinct colors. For \( d > t \geq 3 \), \( G \) contains at most \( t^{d-1}d^d \) vertices of degree at most \( d \).

**Proof.** Let \( X \) be the set of vertices in \( G \) with degree at most \( d \) and assume that \( |X| > t^{d-1}d^d \). We demonstrate the existence of a pair of nonadjacent vertices \( x \) and \( y \) in \( X \) that are
connected by \( d - 1 \) internally disjoint monochromatic paths of length 2. This contradict the assumption that \( x \) and \( y \) are also connected by a pair of internally disjoint rainbow paths.

Observe that \( G \) is connected, \( \Delta(G[X]) \leq d \), and \( |X| \geq d + 2 \); hence \( G[X] \) does not contain \( K_{d+1} \). Therefore Brooks’ Theorem [1] implies that \( G[X] \) is \( d \)-colorable and there is an independent set \( S_0 \subseteq X \) such that \( |S_0| > t^{d-1}d^{d-1} \). For each vertex \( v \in S_0 \) with degree less than \( d \), add \( d - d(v) \) edges joining \( v \) to nonneighbors in \( V(G) - S_0 \). Let \( G' \) be the graph obtained by adding these edges; observe that nonadjacent vertices in \( G' \) are joined by two internally-disjoint rainbow paths of length 2.

We iteratively construct two families of nested sets \( S_0 \supseteq S_1 \supseteq \ldots \supseteq S_{d-1} \) and \( L_0 \subseteq L_1 \subseteq \ldots \subseteq L_{d-1} \) so that \( S_i \) and \( L_i \) satisfy the following four properties for all \( i \in \{0, \ldots, d-1\} \):

1. \( S_i \) is an independent set with \( |S_i| > t^{d-1-i}d^{d-1-i} \),
2. \( |L_i| = i \),
3. every vertex in \( L_i \) adjacent to every vertex in \( S_i \), and
4. if \( x \in L_i \), then every edge joining \( x \) to \( S_i \) has the same color.

Setting \( L_0 = \emptyset \), it is clear that \( S_0 \) and \( L_0 \) satisfy properties (1) through (4) for \( i = 0 \).

Now assume that \( S_i \) and \( L_i \) satisfy properties (1) through (4) for some \( i \in \{0, \ldots, d-2\} \). Pick \( v_i \in S_i \) and let \( N_i = N(v_i) - L_i \). For every \( s \neq v_i \) in \( S_i \), there are two internally disjoint \( s, v_i \)-paths of length 2 in \( G' \). By property (4), every rainbow path of length 2 joining \( v_i \) and \( s \) must contain a vertex in \( N_i \). Therefore \( s \) has at least two neighbors in \( N_i \), and there are at least \( 2(|S_i| - 1) \) edges joining \( S_i \) and \( N_i \). Let \( \ell_{i+1} \) be a vertex in \( N_i \) that is joined to \( S_i \) with the maximum number of edges all of the same color; call this color \( c_{i+1} \). Let \( S_{i+1} \) be the set of vertices in \( S_i \) that are adjacent to \( \ell_{i+1} \) with an edge of color \( c_{i+1} \). By the Pigeonhole Principle,

\[
|S_{i+1}| \geq \frac{2(|S_i| - 1)}{t(d-i)} \geq \frac{|S_i|}{td} > \frac{t^{d-i-1}d^{d-i-1}}{td} = t^{d-(i+1)-1}d^{d-(i+1)-1},
\]

where the second inequality follows from the fact that \( |S_i| \geq 1 \).

Set \( L_{i+1} = L_i \cup \{\ell_{i+1}\} \). Clearly \( L_{i+1} \) satisfies properties (2) and (3). By induction, for every \( \ell \in L_i \) there is a single color on all of the edges joining \( \ell \) and \( S_i \), and consequently \( S_{i+1} \) as well. By definition, all edges joining \( \ell_{i+1} \) and \( S_{i+1} \) have the same color, and hence \( L_{i+1} \) satisfies condition (4).

By Property (1), we know that \( S_{d-1} \) contains at least two vertices. Furthermore, given two vertices \( x \) and \( y \) in \( S_{d-1} \), the \( d-1 \) paths of length 2 joining \( x \) and \( y \) through \( L_{d-1} \) are
all monochromatic. Since \( x \) and \( y \) have degree \( d \) in \( G' \), it follows that there is at most one rainbow path of length 2 joining \( x \) and \( y \) in \( G' \), and consequently also in \( G \).

To complete the proof of Theorem 3, we observe that since every edge in \( F \) lies in a triangle, nonadjacent vertices in any \((\mathcal{R}(F), t)\)-saturated graph must be joined by at least two internally disjoint rainbow paths of length 2. Therefore, Claim 1 applies to all \((\mathcal{R}(F), t)\)-saturated graphs. Let

\[
d = \frac{\log n}{\log \log n}.
\]

By Claim 1, \( G \) contains at most \( t^{d-1}d^d < (td)^d \) vertices of degree at most \( d \). Since \( t \) is a constant, we have that \( (td)^d = (t \log n \log \log n)^d = o(n) \).

It follows that \( G \) has \( n - o(n) \) vertices of degree at least \( d \), and thus

\[
|E(G)| \geq \frac{1}{2}d(n - o(n)) = \Omega \left(n \frac{\log n}{\log \log n}\right).
\]

We now construct an \( n \)-vertex \( \mathcal{R}(F) \)-saturated graph with \( O(n \log n) \) edges when \( F \) is connected and has no independent vertex cut.

**Theorem 4.** Let \( F \) be a \( k \)-vertex graph that is 2-connected and has no independent vertex cut. For \( n \) sufficiently large and \( t \geq |E(F)| \),

\[
sat_t(n, \mathcal{R}(F)) \leq t(k - 2)n \log n - \left(\frac{t(k - 2) \lceil \log n \rceil + 1}{2}\right).
\]

**Proof.** Our goal is to construct an \( n \)-vertex \((\mathcal{R}(F), t)\)-saturated graph \( G \) with a large (on the order of \( n - \log n \)) independent set. Let \( xy \) be an edge of \( F \) such that \( d(x) = \delta(F) \).

To build \( G \), we begin by building a spanning subgraph \( G' \) consisting of many overlapping rainbow copies of \( F - xy \).

For \( i \in [t] \) and \( j \in [\log n] \), let \( S_{i,j} \) be the \((k - 2)\)-vertex set \( \{v^1_{i,j}, \ldots, v^{k-2}_{i,j}\} \). For \( i \in [t] \), let \( S_i = \bigcup_{j=1}^{\log n} S_{i,j} \), and let \( S = \bigcup_{i=1}^t S_i \). Let \( R = \{v_1, \ldots, v_{n-t(k-2)\lceil \log n \rceil}\} \). For each \( i \in [t] \), we will build a graph \( G_i \) on \( S_i \cup R \) such that \( R \) is an independent set and the addition of an edge in color \( i \) joining any two vertices in \( R \) will complete a rainbow copy of \( F \).

For each \( i \in [t] \), let \((F - xy)_i\) be a rainbow-colored copy of \( F - xy \) that does not use color \( i \). For each \( j \in [\lceil \log n \rceil] \), place a copy of \((F - xy)_i - \{x, y\}\) on the vertex set \( S_{i,j} \).
In this section, we determine the Rainbow Saturation Numbers of some specific graphs. To complete $G_t$ we add edges between $S_t$ and $R$ as follows. Assign to each $v_t \in R$ a binary string $(b_{t,1}, \ldots, b_{t,\lceil \log n \rceil})$ of length $\lceil \log n \rceil$ so that the strings are distinct. For each $\ell \in [n - t(k - 2) \lceil \log n \rceil]$, add edges joining $v_\ell$ to $S_{i,j}$ so that

1. if $b_{\ell,j} = 0$, then $S_{i,j} \cup \{v_\ell\}$ induces $(F - xy)_i - \{y\}$, and

2. if $b_{\ell,j} = 1$, then $S_{i,j} \cup \{v_\ell\}$ induces $(F - xy)_i - \{x\}$.

That is, if $b_{\ell,j} = 0$, then $v_\ell$ plays the role of $x$ with respect to $S_{i,j}$; and if $b_{\ell,j} = 1$, then $v_\ell$ plays the role of $y$ with respect to $S_{i,j}$.

To complete the construction of $G'$, we take the union of $G_1, \ldots, G_t$. For distinct $v_\ell, v_{\ell'} \in R$, because the binary strings assigned to $v_\ell$ and $v_{\ell'}$ are distinct, there is a choice of $j$ such that $S_{i,j} \cup \{v_\ell, v_{\ell'}\}$ induces $(F - xy)_i$ for all $i \in [t]$. It follows that the addition of $v_\ell v_{\ell'}$ in any color $i \in [t]$ completes a rainbow copy of $F$ in $G_i$, and consequently in $G'$.

We claim that $G'$ does not contain $F$ as a subgraph, regardless of the edge colors. Recall that the edge $xy$ was chosen so that $d(x) = \delta(F)$, and thus $\delta(F - xy) < \delta(F)$. Therefore if a vertex in $R$ were to be in a copy of $F$, the copy of $F$ would necessarily contain vertices in at least two of the sets $S_{i,j}$. However, the vertices in $R$ form an independent vertex cut that separates $S_{i,j}$ and $S_{i',j'}$ for all distinct pairs $(i, j)$ and $(i', j')$ in $[t] \times [\lceil \log n \rceil]$. Therefore no vertex in $R$ lies in a copy of $F$, and it is clear that no $S_{i,j}$ contains a copy of $F$. Therefore $F$ is not a subgraph of $G'$.

We know that the addition of any edge in any color joining two vertices in $R$ completes a rainbow copy of $F$. However, there may be nonadjacent vertices $u, v \in V(G')$, either both in $S$ or with one in $S$ and one in $R$, and a color $i \in [t]$ such that the addition of $uv$ in color $i$ does not complete a rainbow copy of $F$. If this is the case, then we iteratively add such (colored) edges until no such nonadjacent vertices exist. When no such nonadjacent pairs exist, we have constructed an $(\mathcal{R}(F), t)$-saturated graph; this graph is $G$. Since $R$ is an independent set of size $n - t(k - 2) \lceil \log n \rceil$ in $G$, it follows that $G$ is a subgraph of the complete split graph $K_{t(k - 2) \lfloor \log n \rfloor} \vee K_{n - t(k - 2) \lfloor \log n \rfloor}$, and therefore

$$|E(G)| \leq t(k - 2)n \lceil \log n \rceil - \left(\frac{t(k - 2) \lfloor \log n \rfloor}{2} + 1\right).$$

\[ \square \]

3 Rainbow Saturation Numbers of Some Specific Graphs

In this section, we determine $\text{sat}_t(n, \mathcal{R}(F))$ exactly for several classes of graphs and give upper and lower bounds on others. We begin with a surprising result: there are graphs
whose rainbow saturation numbers are quadratic in $n$. As discussed in the introduction, for any graphs $H_1, \ldots, H_t$, $\text{sat}(n, \mathcal{M}(H_1, \ldots, H_t)) = O(n)$. Hence, the existence of graphs whose rainbow saturation numbers are quadratic in $n$ illuminates an interesting and fundamental difference between the monochromatic and rainbow colored saturation problems.

### 3.1 Stars

Recall that the Kneser graph $K(n, k)$ is the graph with $\binom{n}{k}$ vertices, where each vertex represents a different $k$-subset of $[n]$, and two vertices are adjacent if their corresponding subsets are disjoint. A blow up of a graph is obtained by replacing vertices with independent sets and replacing edges with complete bipartite graphs between the independent sets corresponding to the endpoints of the edge.

**Theorem 5.** If $n \geq (k + 1)(k - 1)/t$, then $\text{sat}_t(n, \mathcal{R}(K_{1,k})) = \Theta\left(\frac{(k-1)}{2t} n^2\right)$.

**Proof.** We begin by characterizing $(\mathcal{R}(K_{1,k}), t)$-saturated graphs. Let $G$ be an $n$-vertex $(\mathcal{R}(K_{1,k}), t)$-saturated graph, and observe first that no vertex is incident to edges of $k$ or more colors, since otherwise $G$ contains a rainbow $K_{1,k}$. Second, if $v$ is incident to edges of at most $k - 2$ colors, then $v$ must have degree $n - 1$. To see this, suppose not, and let $u$ be a nonneighbor of $v$. Choose any edge incident to $u$, of color $c$; adding the edge $uv$ in color $c$ cannot create a rainbow $K_{1,k}$ at $u$, as none previously existed, and also cannot create a rainbow $K_{1,k}$ at $v$, since $v$ sees at most $k - 1$ colors. This contradicts our assumption that $G$ was $(\mathcal{R}(K_{1,k}), t)$-saturated. Finally, observe that if $v$ and $w$ both see color $i$, then $v$ and $w$ must be adjacent; otherwise, adding $vw$ in color $i$ cannot create a rainbow $K_{1,k}$.

By the first two observations, we can partition the vertex set into two parts: A set $Q$ of vertices of degree $n - 1$ that see at most $k - 2$ colors, and a set $A$ of vertices that see exactly $k - 1$ colors. By the third observation, $A$ can be partitioned into $\binom{t}{k-1}$ (possibly empty) cliques according to the set of colors on the edges incident to each vertex. If two such cliques in $A$ correspond to vertices that are incident to edges of a common color, then the cliques must be completely joined. Note that if we contract each clique to a vertex, then the underlying graph formed by $A$ is a subgraph of the complement of the Kneser graph $K(t, k - 1)$. Equivalently, the complement of $G[A]$ is a blow up of a subgraph of $K(t, k - 1)$.

We verify that such a graph is $(\mathcal{R}(K_{1,k}), t)$-saturated. Since no vertex sees $k$ colors, there is no rainbow $K_{1,k}$. The only missing edges are between cliques in $A$ that correspond to vertices that are incident to disjoint color sets. If $v$ and $w$ are in different cliques in $A$ and $vw$ is added in color $i$, then we may assume without loss of generality that $v$ did not
previously have a neighbor of color $i$. Now $v$ is incident to edges with $k$ different colors and hence it is the center of a rainbow $K_{1,k}$.

Having established the structure of $(\mathcal{R}(K_{1,k}), t)$-saturated graphs, in order to find $\text{sat}_t(n, \mathcal{R}(K_{1,k}))$, it suffices to minimize the number of edges in any such graph. Observe that this is equivalent to maximizing the number of edges in the complement of $G$. Since $K(t, k-1)$ does not contain $K_{\lceil t/(k-1) \rceil +1}$, it follows that $\overline{G}$ is $K_{\lceil t/(k-1) \rceil +1}$-free. Therefore, by Turán’s Theorem [10], $\overline{G}$ contains at most $\frac{t/(k-1)-1}{t/(k-1)} \binom{n}{2}$ edges. Thus

$$|E(G)| = \binom{n}{2} - |E(\overline{G})| \geq \left(1 - \frac{t/(k-1)-1}{t/(k-1)}\right) \binom{n}{2} = \frac{k-1}{t} \binom{n}{2}.$$ 

Therefore $\text{sat}_t(n, \mathcal{R}(K_{1,k})) \geq \frac{k-1}{t} \binom{n}{2}$.

Now, let $G$ consist of $\lceil t/(k-1) \rceil$ copies of $K_{n/\lceil t/(k-1) \rceil}$. Edge-color the copies of $K_{n/\lceil t/(k-1) \rceil}$ with pairwise disjoint sets of $k-1$ colors so that each vertex is incident to edges with $k-1$ distinct colors. This graph is $(\mathcal{R}(K_{1,k}, t))$-saturated and has $\Omega\left(\frac{k-1}{t} \binom{n}{2}\right)$ edges.

We note that for specific $k$, $n$, and $t$, the exact value of $\text{sat}_t(n, \mathcal{R}(K_{1,k}))$ can be obtained from the proof of Theorem 5 and Turán’s Theorem. Furthermore, Turán’s Theorem also implies that all $n$-vertex $(\mathcal{R}(K_{1,k}, t))$-saturated graphs of minimum size correspond to edge-colorings of a unique graph.

### 3.2 Paths

We next consider the rainbow saturation number for paths. We begin with a general lower bound on $\text{sat}_t(n, \mathcal{R}(P_k))$, and then show that this bound is correct for $P_4$.

**Proposition 6.** For all $k \geq 4$,

$$\text{sat}_t(n, \mathcal{R}(P_k)) \geq n - 1.$$ 

**Proof.** Let $G$ be an $(\mathcal{R}(P_k), t)$-saturated graph of order $n$. If no component of $G$ is a tree, then $G$ has at least $n$ edges, so we may assume that some component of $G$ is a tree. It suffices to show that $G$ has exactly one such component, so assume by way of contradiction that $T_1$ and $T_2$ are two components of $G$ that are trees.
Let $v$ be a leaf in $T_1$ with neighbor $u$, and suppose the edge $vu$ is color $i$. We claim that the color $i$ cannot appear on any other edge of $T_1$. Suppose otherwise, that $x \in T_1$ is incident to an edge $xy$ of color $i$. Adding the edge $vx$ in color $i$ must create a rainbow $P_k$ (call it $P$) that contains $vx$. Since the color $i$ can be used only once, $P$ must begin $v,x,w,...$, where $w \neq y$. Since $T_1$ is a tree, $y$ cannot appear in $P$, as $P + xy$ would contain a cycle. Hence replacing $vx$ with $yx$ yields a rainbow $P_k$ in $G$, a contradiction.

Now let $v'$ be a leaf in $T_2$ with neighbor $u'$, and assume the edge $v'u'$ is color $j$. Adding the edge $v'u$ in color $j$ must create a rainbow $P_k$. This rainbow $P_k$ must consist of $v'u$ together with a rainbow $P_{k-1}$ beginning at $u$, since color $j$ cannot be used twice. The rainbow $P_{k-1}$ beginning at $u$ must contain and edge of color $i$; otherwise, replacing $v'u$ with $vu$ would yield a rainbow $P_k$. However, this contradicts the fact that color $i$ cannot appear on any edge other than $uv$ in $T_1$. Therefore $G$ contains at most one tree component.

**Corollary 7.** For $t \geq 8$, $\text{sat}_t(n, \mathcal{R}(P_4)) = n - 1$.

**Proof.** The lower bound follows from Proposition 6, so it suffices to construct $(\mathcal{R}(P_4), t)$-saturated graphs of order $n$ with $n - 1$ edges. When $n \equiv 0 \pmod{3}$, a collection of $\lfloor \frac{n}{3} \rfloor$ rainbow triangles and a rainbow clique of order $n \pmod{3}$ suffices. For $n \equiv 0 \pmod{3}$, let $H_n$ be $K_{1,5} + (\frac{n-6}{3})K_3$, colored so that $K_{1,5}$ is rainbow with colors from $[5]$ and each copy of $K_3$ is rainbow with colors from $[t] - [5]$.

### 3.3 Matchings

We conclude this section by considering the rainbow saturation number for matchings. The traditional (uncolored) saturation number of $mK_2$ is $3m$, which appears in [8] and follows immediately from a result of Mader [9] that characterizes $mK_2$-saturated graphs. Similarly, it was shown in [6] that if $m_1, \ldots, m_k \geq 1$ and $n > 3(m_1 + \ldots + m_k - k)$, then

$$\text{sat}(n, \mathcal{R}_{\text{min}}(m_1K_2, \ldots, m_kK_2)) = 3(m_1 + \ldots + m_k - k).$$

While we are unable as yet to determine the rainbow saturation number of $mK_2$ exactly, in line with these results we show next that for $t$ sufficiently large, $\text{sat}_t(n, \mathcal{R}(mK_2)) = O(1)$.

**Theorem 8.** Let $m$ be a positive integer. For $t \geq 5m - 5$ and $n \geq \frac{5}{2}m - 1$,

$$\frac{11}{4}m \leq \text{sat}_t(n, \mathcal{R}(mK_2)) \leq 5m - \epsilon,$$

where $\epsilon = 5$ if $m$ is even and $\epsilon = 4$ if $m$ is odd.
Proof. We begin with the construction for the upper bound. If $m$ is odd, let $G$ be the disjoint union of a rainbow-colored copy of $\frac{m-1}{2} K_5$ and $n - \frac{5(m-1)}{2}$ isolated vertices. If $m$ is even, let $G$ be the disjoint union of a rainbow copy of $\frac{m-2}{2} K_5$ colored from $[5m-5]$, a copy of $K_4$ properly edge-colored from $\{5m-9, 5m-8, 5m-7\}$, and $n - \frac{5(m-2)}{2} - 4$ isolated vertices. In both cases, $G$ is $(\mathcal{R}(mK_2), t)$-saturated and has $rm - \epsilon$ edges, where $\epsilon$ is as defined in the statement of the theorem.

Turning to the upper bound, let $G$ be an $(\mathcal{R}(mK_2), t)$-saturated graph of minimum size. Clearly we may assume that $|E(G)| \leq 5m$. We will use discharging to show that $|E(G)| \geq \frac{11}{4}m$. First we prove various structural properties about $G$.

Claim 2. There is no vertex in $G$ with degree 1.

Proof of Claim 2. Assume that $G$ contains a vertex $u$ of degree 1, let $v$ be the neighbor of $u$, and let $uv$ be colored with color $c$. Since $|E(G)| \leq 5m$, there is a vertex $x$ of degree 0. The addition of $xv$ in color $c$ must create a rainbow $mK_2$. However, substituting $uv$ for $xv$ in this matching yields a rainbow $mK_2$ in $G$, a contradiction.

Claim 3. If $u$ is a vertex of degree 2 in $G$, then the neighbors of $u$ are adjacent.

Proof of Claim 3. Let $N(u) = \{v, x\}$ and assume that $vx \notin E(G)$. Let $uv$ be colored with color $c$. The addition of $vx$ in color $c$ must create a rainbow $mK_2$. However, substituting $uv$ for $xv$ in this matching yields a rainbow $mK_2$ in $G$, a contradiction.

Claim 4. The vertices of degree 2 in $G$ form an independent set.

Proof of Claim 4. Assume that $u$ and $v$ are adjacent vertices of degree 2 in $G$, and let $uv$ be colored with color $c$. By Claim 3, $u$ and $v$ have a common neighbor $x$. Since $|E(G)| \leq 5m$, there is a vertex $y$ of degree 0 in $G$. The addition of $yx$ in color $c$ must create a rainbow $mK_2$. However, substituting $uv$ for $yx$ in this matching yields a rainbow $mK_2$ in $G$, a contradiction.

Claim 5. If $u$ is a vertex of degree 2 with $N(u) = \{v, x\}$ and $d(v) = 3$, then $v$ and $x$ must have two common neighbors.

Proof of Claim 5. Assume to the contrary that $u$ is the only common neighbor of $v$ and $x$. Let $N(v) = \{u, x, v'\}$ and assume that $x$ and $v'$ are not adjacent. Let $uv$ be colored with color $c$. The addition of $xv'$ in color $c$ must create a rainbow $mK_2$. However, substituting $uv$ for $xv'$ in this matching yields a rainbow $mK_2$ in $G$, a contradiction.
Claim 6. No vertex of degree 2 has two neighbors of degree 3.

Proof of Claim 6. Let \( u \) be a vertex of degree 2, let \( N(u) = \{ v, x \} \), and assume that \( d(v) = d(x) = 3 \). By Claim 3, we know that \( vx \in E(G) \). By Claim 5, we know that \( v \) and \( x \) have two common neighbors, \( u \) and \( y \). Let \( vx \) be colored with color \( c \). Since \( d(u) = 2 \), it follows that \( u \) and \( y \) are not adjacent, and therefore the addition of \( uy \) in color \( c \) must create a rainbow \( mK_2 \). However, substituting \( vx \) for \( uy \) in this matching yields a rainbow \( mK_2 \) in \( G \), a contradiction.

At this point, we are ready to describe our discharging rules. Initially, assign every vertex charge equal to its degree. We move charge onto vertices of degree 2 with the following rules. Let \( u \) be a vertex of degree 2, and let \( N(u) = \{ x, y \} \).

1. If \( u \) has a neighbor of degree 3 (without loss of generality, let it be \( x \)), then move charge \( \frac{1}{8} \) from \( x \) to \( u \) and move charge \( \frac{5}{8} \) from \( y \) to \( u \).
2. If \( x \) and \( y \) both have degree at least 4, then move charge \( \frac{3}{8} \) from \( x \) to \( u \) and move charge \( \frac{3}{8} \) from \( y \) to \( u \).

It is clear that after discharging, all vertices of degree 2 have charge \( \frac{11}{4} \). We now show that all other vertices with positive degree have charge at least \( \frac{11}{4} \). If \( d(v) \geq 3 \), then by Claims 3 and 4 it follows that \( v \) has at most \( d(v) - 1 \) neighbors of degree 2. Therefore, if \( d(v) = 3 \), then \( v \) has charge at least \( \frac{11}{4} \) after the discharging rules are applied. If \( v \) has \( d(v) - 1 \) neighbors of degree 3, say \( x_2, \ldots, x_{d(v)} \), then by Claims 3 and 4 it follows that \( x_2 \) is also adjacent to \( x_2, \ldots, x_{d(v)} \). Therefore, if \( v \) has \( d(v) - 1 \) neighbors of degree 2, then \( v \) sends charge \( \frac{3}{8} \) to each of those neighbors and \( v \) has charge at least \( \frac{23}{8} \) after discharging. Otherwise, \( v \) has at most \( d - 2 \) neighbors of degree 2, and therefore \( v \) has charge at least \( d(v) - \frac{5}{8}d(v) - 2 = \frac{3}{8}d(v) + \frac{5}{4} \geq \frac{11}{4} \) after discharging.

Since \( G \) is \( (\mathcal{R}(mK_2), t) \)-saturated, there must be at least \( 2m \) vertices of positive degree. Otherwise, either (a) \( G \) contains \( K_{2m-1} \) and hence \( |E(G)| \geq \binom{2m-1}{2} > \frac{11}{4}m \), or (b) it is possible to add an edge to \( G \) and have at most \( 2m - 1 \) vertices of positive degree. From the discharging process, we know that the vertices of positive degree have average degree at least \( \frac{11}{4} \). Therefore \( |E(G)| \geq \frac{11}{4}m \).

4 Upper bounds

In this section we prove upper bounds on rainbow saturation numbers with constructions of \( (\mathcal{R}(F), t) \)-saturated graphs for various choices of \( F \). First we prove that the rainbow
saturation number for most trees is linear in $n$.

**Theorem 9.** Let $H$ be a connected $k$-vertex graph with $k \geq 5$. If $H$ has a vertex $v$ with $d(v) = 1$ whose neighbor $v'$ does not have degree $k - 1$, there are two vertices $u$ and $u'$ in $V(H) \setminus \{v, v' \}$ that are not adjacent, and $t \geq \binom{k-1}{2}$, then

$$\text{sat}_t(n, \mathcal{R}(H)) \leq \left\lceil \frac{n}{k - 1} \right\rceil \binom{k - 1}{2}.$$

In particular, if $T$ is a tree with at least four vertices that is not a star, then

$$\text{sat}_t(n, \mathcal{R}(T)) = O(n).$$

**Proof.** Let $G$ be the $n$-vertex graph formed by partitioning the vertex set into $\left\lfloor \frac{n}{k - 1} \right\rfloor$ cliques of size $k - 1$ and one clique containing the remaining $n \mod (k - 1)$ vertices. Rainbow color the edges of each clique. We claim that $G$ is $(\mathcal{R}(H), t)$-saturated. Suppose that the edge $xy$ is added in color $i$, where $x$ is in the clique $G_x \subseteq G$ and $y$ is in the clique $G_y \subseteq G$. Assume by symmetry that $G_x$ has $k - 1$ vertices; hence $G_x$ contains a rainbow copy of every $k - 1$-vertex graph. We claim that $G_x$ contains a rainbow copy of $H - v$ avoiding the color $i$. Set $x = v'$. Since $G_x$ is rainbow-colored, $G_x$ contains at most one edge $ab$ with color $i$. If $x = a$, then let $b$ be a nonneighbor of $v'$ in $H$. If $x \neq a, b$, then let $a = u$ and $b = u'$. All other vertices of $H - v$ can be assigned arbitrarily to vertices of $G_x$, and we obtain the needed rainbow $H - v$ avoiding the color $i$. Now this together with the edge $xy$ in color $i$ creates a rainbow copy of $H$.

Finally, if $T$ is a tree that is not a star, then either $T$ has order at least 5 or $T = P_4$. Therefore, this construction and Proposition 6 together imply that $\text{sat}_t(n, \mathcal{R}(T)) = O(n)$. \qed

Our next construction also provides a linear bound on rainbow saturation numbers. This construction is a natural analogue to the construction used for Theorem 4, and it too applies to graphs that have no independent vertex cut. However, it also applies to graphs with girth at least 4 in which most of the cycles of minimum length share a common edge. In particular, this family includes all cycles of length at least 4.

**Theorem 10.** Let $F$ be a $k$-vertex connected graph satisfying one of the following properties:

1. there is an edge in $F$ that does not lie on a triangle, $F$ is 2-connected, and $F$ does not have an independent vertex cut;

2. $F$ has girth at least 4 and more than half of the cycles of minimum length in $F$ contain a common edge.
It follows that \( \text{sat}_t(n, \mathcal{R}(F)) \leq 2(k - 2)n - \binom{2k - 3}{2} \). Therefore \( \text{sat}_t(n, \mathcal{R}(F)) = O(n) \).

Proof. We describe an \( n \)-vertex graph \( G \) with \( O(n) \) edges that is \((\mathcal{R}(F), t)\)-saturated. We begin by describing a subgraph \( G' \) of \( G \). If \( F \) satisfies Condition 1 of the statement of the theorem, let \( xy \) be an edge in \( F \) that does not lie on a triangle. If \( F \) satisfies Condition 2, let \( xy \) be an edge that is shared by more than half of the cycles of minimum length in \( G \).

To create \( G' \), let \( F_1 \) and \( F_2 \) be two copies of \( F - xy \). If \( u \in V(F) \), then denote by \( u_i \) the copy of \( u \) in \( F_i \). Let \( c \) be a rainbow coloring of the edges of \( F_1 \) and \( F_2 \) so that no color is reused. Create an independent set \( \mathcal{V} \) of size \( n - 2k \), and let \( R \) be the independent set together with \( x_1, x_2, y_1, \) and \( y_2 \). For every neighbor \( a_i \) of \( x_i \) in \( F_i \), add an edge of color \( c(x_i, a_i) \) connecting \( a_i \) to each vertex in \( R \). Similarly, for every neighbor \( b_i \) of \( y_i \) in \( F_i \), add an edge of color \( c(v_i, y_i) \) connecting \( y_i \) to each vertex in \( R \). Since \( x \) and \( y \) have no common neighbors, this coloring is well defined.

Notice first that adding an edge in any color between vertices of \( R \) will create a rainbow copy of \( F \). If the edge \( uv \) is added to \( R \) in color \( j \), since the color sets of \( F_1 \) and \( F_2 \) are disjoint, then \( (V(F_i) \cup \{u, v\}) - \{x_i, y_i\} \) spans a rainbow copy of \( F \) for some \( i \in \{1, 2\} \).

We claim that \( G' \) contains no rainbow copy of \( F \). Observe that \( G'[V(F_i) \cup R] \) contains only \( |E(F)| - 1 \) edge-colors for \( i \in \{1, 2\} \). Therefore, any rainbow copy of \( F \) in \( G' \) would have to contain edges from both \( F_1 \) and \( F_2 \). However, if \( F \) satisfies Condition 1 of the theorem, then such a copy of \( F \) would contain an independent vertex cut in \( R \). Now suppose that \( F \) satisfies Condition 2 of the theorem. Note that any cycle that contains vertices in both \( F_1 \) and \( F_2 \) necessarily contains at least \( 2(\text{girth}(F) - 1) \) edges. Therefore every cycle of length \( \text{girth}(F) \) in \( G' \) lies in \( F_1 \) or \( F_2 \). Since \( xy \) lies on more than half of the cycles of minimum length in \( F \), it follows that no edge in \( F_1 \) or \( F_2 \) lies on that many cycles of length \( \text{girth}(F) \). Therefore no rainbow copy of \( F \) appears in \( G' \).

Note that \( G' \) may not be \((\mathcal{R}(F), t)\)-saturated, since there may be nonadjacent vertices \( u, v \in V(G') \), either both in \( V(G') \setminus R \) or with one in \( V(G) \setminus R \) and one in \( R \), and a color \( i \in [t] \) such that the addition of \( uv \) in color \( i \) does not complete a rainbow copy of \( F \). If this is the case, then we iteratively add such (colored) edges until no such nonadjacent vertices exist. When no such nonadjacent pairs exist, we have constructed an \((\mathcal{R}(F), t)\)-saturated graph; this graph is \( G \). Since \( R \) is an independent set of size \( n - 2(k - 2) \) in \( G \), it follows that \( G \) is a subgraph of the complete split graph \( K_{2(k - 2)} \vee \overline{K}_{n - 2(k - 2)} \), and therefore

\[
|E(G)| \leq 2(k - 2)n - \binom{2k - 3}{2}.
\]

\( \square \)
5 Conclusions and Open Problems

There are many open questions remaining regarding rainbow saturation numbers of graphs. We conjecture that the upper bound given by Theorem 4 for complete graphs is the correct order of growth for the rainbow saturation number of complete graphs.

Conjecture 2. sat_{t}(n, R(K_k)) = \Theta(n \log n).

Another clear direction for future work is to close the gap between the lower and upper bounds in Theorem 5 to obtain the asymptotic, if not exact, value.

Question 1. What is the (asymptotic) value of sat_{t}(n, R(mK_2))?

The quadratic value of sat_{t}(n, R(K_{1,k})) from Theorem 5 depends strongly on the fact that adding an edge to an (R(K_{1,k}), t)-saturated graph must increase the number of colors on the edges incident to some vertex in the graph. The next question asks if this is the only property that will lead to quadratic rainbow saturation numbers.

Question 2. Is there a graph G that is not a star such that sat_{t}(n, R(G)) = \Theta(n^2)?

Throughout this paper, we have taken the number of colors available to be a sufficiently large number. There is certainly a host of compelling questions to consider when the number of colors available is small. Observe that sat_{t}(n, R(F)) is nonincreasing in t since every (R(F), t)-saturated graph is also (R(F), t + 1)-saturated. This motivates the following general question.

Question 3. How does sat_{t}(n, R(F)) grow as t approaches |E(F)| from above?

References


