1 Vectors

Throughout physics we encounter different types of quantities including scalars (mass, temperature, charge), vectors (force, acceleration, electric field), and tensors (moment of inertia, stress) although you have probably not had to deal with tensors much so far. Let's review the first two, introducing the notation used in the text (and my lectures).

1.1 Scalars and notation

Fowles states in Section 1.3 “A scalar is a physical quantity that has magnitude only...” This can be misleading, since using the word “magnitude” suggests an absolute magnitude that must be positive, while scalars can also be negative. Here is my definition.

A scalar is a physical quantity that has a sign, a magnitude, and often associated units. Thus \( \pi \) is a positive scalar with no units, but the charge of an electron, \(-1.6 \times 10^{-19} \text{ C} = -4.8 \times 10^{-10} \text{ statcoulomb}\), is a negative scalar with units.

Scalars follow all the usual algebraic rules of addition, subtraction, multiplication, division, etc.

In print, ordinary italic symbols are used for scalars such as \( m, T, q \). In handwriting of course we make no effort to distinguish italic from non-italic symbols.

Here is another way to think of scalars. At various points in the text we will switch to other coordinate systems, such as a rotated coordinate system. The value of a scalar remains unchanged when we switch to a new coordinate system.

So if an object has a mass of 2.0 kg, a speed of 5.0 m/s, a velocity in the \( x \)-direction, if we look in a coordinate system rotated by \(-37^\circ\) the mass and speed are still 2.0 kg and 5.0 m/s but the velocity has components 4.0 m/s in the \( x \) and 3.0 m/s in the \( y \).
1.2 Vectors in 2D, Cartesian and Polar representations, pictorial

The modern notion of vectors arose in the late 19th century, through the work of Gibbs (Gibb’s Free Energy in thermodynamics) and Heaviside (the Heaviside function in mathematics.)

I paraphrase Fowles. A vector has magnitude, direction, and possibly units, with the magnitude really meaning absolute magnitude for our work, and the direction given typically by some angles relative to a coordinate system.

Various notations are used for vector symbols. In print boldface symbols are common, \( \mathbf{v} \) (either italic or non-italic), in handwriting \( \vec{v} \) is convenient although some use the typesetter’s convention of an underscore to indicate boldface, \( \vec{v} \). The magnitude of a vector can be written in two ways, \( |\vec{v}| = v \).

Consider two-dimensional space. A 2D vector can be given numeric values once we establish a coordinate system (\( x-y \) let’s say) either in plane polar or cartesian coordinates. We assume that the tail of the vector is at the origin of the coordinate system and want to specify the location of the tip.

In plane polar we give the magnitude of the vector and its units, and an angle measured from one of the axes. Thus we could have a vector \( \vec{v} = (+3.3 \text{ m in a direction } 127^\circ \text{ counterclockwise from the } x\text{-axis.}) \) By convention we use positive angles for counterclockwise, and unless otherwise stated assume that we are measuring from the \( x\)-axis. It is nice, but not necessary, to keep the angle limited to \( -\pi < \theta < +\pi \). In symbols we could write \( \vec{V} = (V, \theta) \).

When we switch to a different reference system, such as a rotated reference system, the numbers needed to specify the vector must transform. In the polar 2D situation, when we switch to a rotated coordinate system the magnitude (a scalar) remains the same but the angle changes.

Alternately we could look at the vector components along the two axes, \( \vec{V} = (V_x, V_y) \). The components are signed scalars with units.

Converting between these two representations is easy:

\[
V_x = V \cos \theta \\
V_y = V \sin \theta
\]

(1)

and

\[
V = \sqrt{V_x^2 + V_y^2} \\
\theta = \tan^{-1} \frac{V_y}{V_x}
\]

(2)
Of course one problem remains—the arctan returns angles between \(-\pi/2 < \theta < \pi/2\) and therefore is wrong half the time. You should already know how to remedy this. Many calculators and spreadsheets have an alternate function “atan2\((V_y, V_x)\)” or “atan2\((V_x, V_y)\)” that will return the correct angle.

We can define the angle between the \(y\)-axis and the vector, \(\phi = \theta - \pi/2\). Then we can write Equation 1 as

\[
\begin{align*}
V_x &= V \cos \theta \\
V_y &= V \cos \phi
\end{align*}
\]

The angles \(\theta, \phi\) are called the \textit{direction angles}, and the trig function values, the \textit{direction cosines}. We will discuss these in 3D shortly.

Vectors must also obey the rules of vector algebra including the rules for addition and multiplication. Vector addition is \textit{commutative} and \textit{associative}. Not all directed quantities do this as we will demonstrate with large rotations.

It is convenient to introduce unit vectors. These are vectors with a magnitude of 1 and no units. In general Fowles uses \(e_r \equiv \hat{e}_r\) for a unit vector in the \(r\)-direction\(^1\). For the cartesian system \(\hat{i}, \hat{j}, \hat{k}\) are used\(^2\). We can then write the 2D vector as

\[
\vec{V} = V_x \hat{i} + V_y \hat{j}
\]

In Section 1.3, Fowles gives some of the basic relationships for vectors in terms of their components, including addition, subtraction, multiplication by a scalar, commutative and associative laws for addition, and distributive law for multiplication of a vector sum by a scalar, and magnitude.

Suppose that the tail of the vector is not at the origin of the coordinate system. How do we determine the vector? We can introduce two vectors that locate the tail and tip of the vector relative to the origin, call these \(\vec{V}_T, \vec{V}_H\) and then \(\vec{V} = \vec{V}_H - \vec{V}_T\).

### 1.3 Vectors in 3D, Product of Vector and Scalar

When we go to three dimensions, we need three numbers to specify a vector. In cartesian representation this will just be

\[
\vec{V} = (V_x, V_y, V_z) = V_x \hat{i} + V_y \hat{j} + V_z \hat{k}
\]

\(^1\)Other notation includes \(u_r\), etc. and \(\hat{r}, \hat{\theta}, \hat{\phi}\).

\(^2\)Other authors use \(\hat{x}, \hat{y}, \hat{z}\).
Alternately we can give a magnitude and two angles, \((V, \theta, \phi)\)—we will describe spheri-
cal polar coordinates shortly—or two magnitudes and one angle, \(r, \theta, z\)—as in cylindrical
coordinates. Other choices are possible.

Rules of addition and subtraction are simple extensions of the 2D case. Multiplication
of a vector by a scalar can be written as
\[
c\vec{V} = (cV_x, cV_y, cV_z) = (cV, \theta, \phi)
\]
however in the polar form we will need to discuss the case of negative scalars carefully—that
will come soon.

### 1.4 Dot product aka scalar or inner product, 2D, 3D

There are several ways to define multiplication between two vectors. For the present we
will use dot (scalar) and cross (vector) products.\(^3\)

Define the dot product between vectors as
\[
\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z \quad \text{in 3D}
\]
\[
\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y \quad \text{in 2D}
\]

A simple geometric argument in 2D lets us see that the dot product in 2D is also
\[
\vec{A} \cdot \vec{B} = AB \cos \theta
\]
where \(\theta\) is the angle between the two vectors.

The dot product can be considered as the magnitude of \(\vec{A}\) multiplied by the projection of
\(\vec{B}\) onto \(\vec{A}\), or the magnitude of \(\vec{B}\) multiplied by the projection of \(\vec{A}\) onto \(\vec{B}\).

We can extend this to 3D. Given \(\vec{A} = A_x \hat{i} + A_y \hat{j} + A_z \hat{k}\), then \(\vec{A} \cdot \hat{i} = A(1) \cos \alpha = A_x\)
where \(\alpha\) is the angle between the vector and the \(x\)-axis, and \(\cos \alpha\) is the direction cosine.

Dot products with \(\hat{j}\) and \(\hat{k}\) give direction angles \(\beta\) and \(\gamma\), with only two of these angles
being independent. **These direction angles are not equal to the angles found in spherical
polar representation as we will show later.**

The direction cosines define a unit vector
\[
\hat{n} = \hat{i} \cos \alpha + \hat{j} \cos \beta + \hat{k} \cos \gamma
\]
so that we can write \(\vec{A} = A\hat{n}\).

Note that \(\hat{n} \cdot \hat{n} = 1\) and this constrains possible values of the direction angles.

\(^3\)There are also exterior products, wedge products, and outer products. Don’t ask!
Examples 1.4.1 to 1.4.4 give important applications of the dot product.

Dot products are commutative. Since the dot product is only defined for two vectors at a time, no associative property is needed.

Do either of the following have meaning?

\[ \vec{A} \cdot \vec{B} \cdot \vec{C} \]
\[ \vec{A} \left( \vec{B} \cdot \vec{C} \right) \]

### 1.5 Cross Product (vector product)

Providing we are in 3D, we can define a product of two vectors that results in a third vector. In cartesian representation, define \( \vec{A} \times \vec{B} \) as

\[ \vec{A} \times \vec{B} = (A_y B_z - A_z B_y, A_z B_x - A_x B_z, A_x B_y - A_y B_x) \] (10)

Notice the cyclic nature of the result: the \( x \)-term stars with \( A_y B_z \), the \( y \)-term with \( A_z B_x \), and the \( z \)-term with \( A_x B_y \) and the subscripts come in order.

In terms of a matrix determinant

\[ \vec{A} \times \vec{B} = \text{det} \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{pmatrix} \] (11)

The cross product is anticommutative,

\[ \vec{B} \times \vec{A} = -\vec{A} \times \vec{B} \] (12)

distributes across addition,

\[ \vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C} \] (13)

and follows

\[ n \left( \vec{A} \times \vec{B} \right) = \left( n \vec{A} \right) \times \vec{B} = \vec{A} \times \left( n \vec{B} \right) \] (14)

Fowles show that the magnitude of the cross product is just \( AB \sin \theta \). The direction is given by the right-hand rule.

Look carefully at Examples 1.5.1 to 1.5.3 in Fowles.
1.6 Triple Products

How can we combine dot and cross products into meaningful three-vector expressions?

Earlier you should have concluded that $\vec{A} \cdot \vec{B} \cdot \vec{C}$ has no meaning. Likewise $\vec{A} \times \left( \vec{B} \cdot \vec{C} \right)$ has no meaning. The other two combinations have meanings and regularly arise.

The scalar triple product is

$$\vec{A} \cdot \left( \vec{B} \times \vec{C} \right) = \det \begin{pmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{pmatrix}$$

(15)

and we can show

$$\vec{A} \cdot \left( \vec{B} \times \vec{C} \right) = \left( \vec{A} \times \vec{B} \right) \cdot \vec{C}$$

(16)

You should be able to show that the vector triple product is the following

$$\vec{A} \times \left( \vec{B} \times \vec{C} \right) = \vec{B} \left( \vec{A} \cdot \vec{C} \right) - \vec{C} \left( \vec{A} \cdot \vec{B} \right)$$

(17)

and Fowles suggests the mnemonic “BAC minus CAB.” This will be used when we discuss rotating coordinate systems in Chapter 5 and rigid body rotation in Chapter 9.

2 Coordinate Change Between Rotated Systems

Consider two coordinate systems that share the same origin but are rotated with respect to each other. This will arise when we talk about motion in a rotating coordinate frame and when we discuss rigid body rotation.

We use primes for the second coordinate system and can write

$$\vec{A} = \hat{i}A_x + \hat{j}A_y + \hat{k}A_z$$

$$= \hat{i}'A'_{x'} + \hat{j}'A'_{y'} + \hat{k}'A'_{z'}$$

(18)

Recognizing that

$$A_x = \vec{A} \cdot \hat{i}$$

$$A'_{x'} = \vec{A} \cdot \hat{i}'$$

(19)

\textsuperscript{4}Recall that if matrix $Y$ is obtained from matrix $X$ by the interchange of two rows (or columns), then $\det(Y) = -\det(X)$. Other properties of matrices are summarized in Appendix H of Fowles & Cassiday.
it is easy to show that we can write a matrix representation for the transformation,

\[
\begin{pmatrix}
A' \\
B' \\
C'
\end{pmatrix} =
\begin{pmatrix}
\hat{i} \cdot \hat{i}' & \hat{j} \cdot \hat{j}' & \hat{k} \cdot \hat{k}' \\
\hat{i} \cdot \hat{k}' & \hat{j} \cdot \hat{j}' & \hat{k} \cdot \hat{k}' \\
\hat{i} \cdot \hat{j}' & \hat{j} \cdot \hat{k}' & \hat{k} \cdot \hat{k}'
\end{pmatrix}
\begin{pmatrix}
A \\
B \\
C
\end{pmatrix}
\]

(20)

E.g. 1 Show that the rotation matrix for a 37° rotation about the x-axis is

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 0.8 & 0.6 \\
0 & -0.6 & 0.8
\end{pmatrix}
\]

E.g. 2 Show that the rotation matrix for a 53° rotation about the y-axis is

\[
\begin{pmatrix}
0.6 & 0 & -0.8 \\
0 & 1 & 0 \\
0.8 & 0 & 0.6
\end{pmatrix}
\]

E.g. 3 Determine the rotation matrix for a rotation of 37° about the x-axis followed by a rotation of 53° about the y-axis. Find the rotation matrix if the order of the rotations is reversed, and notice that the results are different, i.e. that rotation is non-commutative.

3 Derivatives of Vectors

Consider a vector that is a function of a variable \(u\). We can apply the normal product rule of calculus to write

\[
\frac{d(n\vec{A})}{du} = \frac{dn}{du}\vec{A} + n\frac{d\vec{A}}{du}
\]

(21)

\[
\frac{d(\vec{A} \cdot \vec{B})}{du} = \frac{d\vec{A}}{du} \cdot \vec{B} + \vec{A} \cdot \frac{d\vec{B}}{du}
\]

(22)

\[
\frac{d(\vec{A} \times \vec{B})}{du} = \frac{d\vec{A}}{du} \times \vec{B} + \vec{A} \times \frac{d\vec{B}}{du}
\]

(23)

Considering a vector \(\vec{A} = \hat{i}A_x\), we can write \(d\vec{A}/du = (dA_x/du)\hat{i} + A_x\left(\hat{i}/du\right)\), but since the unit vectors in an inertial set of cartesian coordinates are constant in magnitude and direction, derivatives of the unit vectors are zero. So in general for inertial cartesian coordinates\(^5\) we write

\[
\frac{d\vec{A}}{du} = \hat{i} \frac{dA_x}{du} + \hat{j} \frac{dA_y}{du} + \hat{k} \frac{dA_z}{du}
\]

(24)

\(^5\)Later in the course we will use a cartesian set of coordinates that is rotating relative to the inertial set. For the rotating coordinates the derivatives of unit vectors will not be zero.
4 Derivatives of Vectors in Plane Polar Coordinates

If we have a situation where an object is moving in a circle, or a spiral, it may suggest using plane polar coordinates \((r, \theta)\).

Multiplying this vector by a negative scalar \(-|c|\) to get \((-|c|r, \theta)\) would imply a negative magnitude, and that violates our idea of the magnitude as being positive. We prefer to switch the direction to get \((|c|r, \theta + \pi)\).

It is convenient to define two unit vectors for this coordinate system, \(\hat{e}_r, \hat{e}_\theta\). These have magnitude 1, and directions that correspond to increasing values of the coordinate. Thus \(\hat{e}_r\) is away from the origin, and by the usual convention, \(\hat{e}_\theta\) is in the counterclockwise sense.

The location of a particle can be written
\[
\vec{r} = r\hat{e}_r
\]  
but unlike the cartesian case, the unit vector changes direction depending on where the particle is located. So if we want to find the velocity of the particle we must evaluate
\[
\vec{v} = \frac{d\vec{r}}{dt} = \frac{dr}{dt}\hat{e}_r + r\frac{d\hat{e}_r}{dt}
\]

**Notation:** derivatives with respect to time will be indicated by dots, \(v = \dot{r}, a = \ddot{r}, \dot{v} = \ddot{r}\).

Fowles has a great geometrical argument showing how to evaluate the unit vector derivative. Here is an alternate approach. Write \(\hat{e}_r = \cos \theta \hat{i} + \sin \theta \hat{j}\) and doing the derivative of this
\[
\frac{d\hat{e}_r}{dt} = \dot{\theta} \left( -\sin \theta \hat{i} + \cos \theta \hat{j} \right)
\]
The dot product \(\hat{e}_r \cdot \frac{d\hat{e}_r}{dt} = 0\), meaning the derivative is perpendicular to \(\hat{e}\), and from the signs of the components it must be in quadrant 2 (negative \(x\), positive \(y\)), hence in the direction of increasing angle. The quantity in parentheses has magnitude 1, and thus is a unit vector. Hence
\[
\frac{d\hat{e}_r}{dt} = \dot{\theta} \hat{e}_\theta
\]
Likewise
\[
\frac{d\hat{e}_\theta}{dt} = -\dot{\theta} \hat{e}_r
\]
Returning to Equation 26, we can write it as
\[
\vec{v} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta
\]
So if we move in a circle of constant radius, $\dot{r} = 0$, the velocity must be tangential to the circle. However if we are looking at a particle in a centrifuge or motion along a coil we have both radial and tangential components of velocity.

Taking another time derivative, we get the acceleration and you should easily be able to show

$$\vec{a} = \left( \ddot{r} - r \dot{\theta}^2 \right) \hat{e}_r + \left( r \ddot{\theta} + 2 \dot{r} \dot{\theta} \right) \hat{e}_\theta$$  \hspace{1cm} (31)

5 Derivatives of Vectors in Cylindrical Coordinates

Some situations have cylindrical symmetry, such as the flow of a liquid through a pipe. By using cylindrical coordinates rather than cartesian coordinates the solutions become easier.

The cylindrical coordinates are $(R, \phi, z)$ where we choose $z$ by convention—we could as easily choose $x$ or $y$. This is an extension of plane polar into the third dimension, and we have

$$
x = R \cos \phi \\
y = R \sin \phi \\
z = z$$  \hspace{1cm} (32)

and (with the usual ambiguity in the inverse tangent)

$$R = \sqrt{x^2 + y^2} \\
\phi = \tan^{-1} \frac{y}{x} \\
z = z$$  \hspace{1cm} (33)

Multiplying by a negative scalar results in $-|c|(R, \phi, z) = (|c| R, \phi + \pi, -|c|z)$.

The unit vectors are $\hat{e}_R, \hat{e}_\phi, \hat{e}_z = \hat{k}$. The unit vectors must make a right handed triad, i.e. $\hat{e}_R \times \hat{e}_\phi = \hat{k}$ and cyclic permutations.

The position vector is $\vec{r} = R \hat{e}_R + z \hat{k}$.

The derivatives of the unit vectors are

$$
\frac{d\hat{e}_R}{dt} = \dot{\phi} \hat{e}_\phi \\
\frac{d\hat{e}_\phi}{dt} = -\dot{\phi} \hat{e}_R \\
\frac{dk}{dt} = 0
$$  \hspace{1cm} (34)
Using these we can easily determine the velocity and acceleration

\[ \vec{v} = \dot{R} \hat{e}_R + R \dot{\phi} \hat{e}_\phi + \ddot{z} \hat{k} \]  

\[ \vec{a} = \left( \ddot{R} - R \dot{\phi}^2 \right) \hat{e}_R + \left( \dddot{R} + 2 \ddot{R} \dot{\phi} \right) \hat{e}_\phi + \dddot{z} \hat{k} \]  

6 Derivatives of Vectors in Spherical Polar Coordinates

In the case of spherical symmetry, spherical polar coordinates are often the best choice. In physics and engineering\(^6\) we have coordinates \((r, \theta, \phi)\) where \(\theta\) is the polar angle measured (by convention) from the z-axis, \(0 < \theta < \pi\), and \(\phi\) is the azimuthal angle measured between the x-axis and the projection of the vector onto the x-y plane, \(-\pi < \phi < \pi\) or \(0 < \phi < 2\pi\).

The position is given by \(\vec{r} = r \hat{e}_r\).

The transformations between cartesian and spherical polar are

\[
\begin{align*}
    z &= r \cos \theta \\
    x &= r \sin \theta \cos \phi \\
    y &= r \sin \theta \sin \phi 
\end{align*}
\]  

and (the inverse cosine for \(\theta\) will return an angle between 0 and \(\pi\) as desired, but the inverse tangent for \(\phi\) is ambiguous)

\[
\begin{align*}
    r &= \sqrt{x^2 + y^2 + z^2} \\
    \theta &= \cos^{-1} \frac{z}{r} \\
    \phi &= \tan^{-1} \frac{y}{x} 
\end{align*}
\]  

Multiplying by a negative scalar results in \(-|c|(r, \theta, \phi) = (|c|r, \pi - \theta, \phi + \pi)\).

We also need to find derivatives of the unit vectors with time. Start by writing expressions for the unit vectors—this requires careful thinking about the unit vectors as is discussed in Section 1.12 of Fowles. The result is

\[
\begin{align*}
    \hat{e}_r &= \hat{i} \sin \theta \cos \phi + \hat{j} \sin \theta \sin \phi + \hat{k} \cos \theta \\
    \hat{e}_\theta &= \hat{i} \cos \theta \cos \phi + \hat{j} \cos \theta \sin \phi - \hat{k} \sin \theta \\
    \hat{e}_\phi &= -\hat{i} \sin \phi + \hat{j} \cos \phi
\end{align*}
\]  

\(^6\)In math the angles are typically reversed, with \(\phi\) being the polar angle. Math may also use \(\rho\) rather than \(r\).
Next by doing derivatives using the product rule, and collecting terms in $\left( {\hat{\theta}}, {\hat{\phi}} \right)$ we can get

$$
\begin{align*}
\frac{d{\hat{e}}_r}{dt} &= {\hat{e}}_\theta {\hat{\theta}} + {\hat{e}}_\phi {\hat{\phi}} \sin \theta \\
\frac{d{\hat{e}}_\theta}{dt} &= -{\hat{e}}_r {\hat{\theta}} + {\hat{e}}_\phi {\hat{\phi}} \cos \theta \\
\frac{d{\hat{e}}_\phi}{dt} &= -{\hat{e}}_r \sin \theta - {\hat{e}}_\theta \cos \theta
\end{align*}
$$

(F40)

Fowles has an error in his Eqn 1.12.11c. My expression, the final equation in (40) is correct.

The velocity and acceleration then can be determined relatively easily.

$$
\vec{v} = {\hat{e}}_r \dot{r} + {\hat{e}}_\theta r \dot{\theta} + {\hat{e}}_\phi r \dot{\phi} \sin \theta 
$$

(41)

$$
\vec{a} = {\hat{e}}_r \left( \ddot{r} - r \dot{\phi}^2 \sin^2 \theta - r \dot{\theta}^2 \right) + {\hat{e}}_\theta \left( r \ddot{\theta} + 2 \dot{r} \dot{\theta} - r \dot{\phi}^2 \sin \theta \cos \theta \right) + {\hat{e}}_\phi \left( r \ddot{\phi} \sin \theta + 2 \dot{r} \dot{\phi} \sin \theta + 2 r \dot{\theta} \dot{\phi} \cos \theta \right)
$$

(42)

7 Instantaneous Orthogonal System

Suppose that a particle follows some complicated curved trajectory in 3D. At any point in its motion we can define a coordinate system consisting of a unit vector tangent to the trajectory and two normal unit vectors.

The tangent unit vector, $\hat{\tau}$ is tangent to the trajectory and in the direction of motion.

Considering points on the trajectory close to the point of interest, we can deduce that for small enough intervals, the trajectory lies on some plane, and three points on this trajectory will allow us to determine the center of curvature of the small arc and the radius of curvature, $\rho$. The first normal, $\hat{n}_1$, will lie in this plane and point away from the center of curvature. The second normal, $\hat{n}_2$, will be perpendicular to both the tangent and first normal and will form a right handed coordinate system with them.

8 Existence of Other Orthogonal Coordinate Systems

Much of physics is done in one of the the coordinate systems discussed above: 2D and 3D Cartesian, Plane Polar, Cylindrical, and Plane Polar. There exist several other orthogonal coordinate systems that are useful when a problem has a specific symmetry. Details are left for advanced courses in Mechanics or Mathematical Physics.
Table 1: Some Named 2D Orthogonal Systems

<table>
<thead>
<tr>
<th>System</th>
<th>Cartesian</th>
<th>Parabolic</th>
<th>Two Center Bipolar</th>
<th>Plane Polar</th>
<th>Bipolar</th>
<th>Hyperbolic</th>
<th>Biangular</th>
<th>Elliptic</th>
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Table 2: Some Named 3D Orthogonal Systems

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<th>Bipolar Cylindrical</th>
<th>Cylindrical</th>
<th>Prolate Spheroidal</th>
<th>Conical</th>
<th>Spherical Polar</th>
<th>Ellipsoidal</th>
<th>Flat-ring cyclide</th>
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