Chapter 10 Notes: Lagrangian Mechanics

January 5, 2011

Thus far we have solved problems by using Newton's Laws (a vector approach) or energy conservation (a scalar approach.) In this chapter we approach the subject in a very different fashion, and one that initially seems far from evident. We will still use kinetic and potential energies, but will define the Lagrangian, $L = T - V$, from which we can generate the equations of motion by doing simple derivatives

The power of this approach will become evident in examples. The development of this approach spanned the work of Wilhelm von Leibniz[1](1646-1716), Johann Bernoulli[2] (1667-1748), Jean LeRond D'Alembert[3] (1717-1783), and Joseph Louis de Lagrange[4] (1736-1813). We begin with William Rowan Hamilton's[5] (1805-1865) Variational Principle espoused after the work of those previously mentioned.

When the Lagrangian method works it is very slick. When non-conservative forces are present Newton's Laws may still be preferred.

1 Hamilton's Variational Principle

Consider a conservative system. Define the *Lagrangian* of a system as

$$L = T - V \tag{1}$$

the difference between kinetic and potential energies. To do this we must already know expressions for the energies. Translational kinetic energy is simply $T = \frac{1}{2}mv^2$ and potential energies are found from the work done by a conservative force.

[1] A German, Leibniz was one of the inventors of calculus. And also worked on mechanical calculators.

[2] One of a large family prominent in math and physics, Johann worked on the newly discovered calculus. From Switzerland, Johann fought first with his brother Jakob, then with his son Daniel.

[3] French, created ratio test for convergence of a series.

[4] Italian, major force in development of the calculus of variations.

[5] Irish, developed Hamiltonian Mechanics, widely used in quantum mechanics.

To make graphs simple we will consider 1D motion: the results are true in 3D as well. Suppose that the system evolves from time t_1 to t_2 along a particular trajectory, that is a particular $y(t), \dot{y}(t)$. The starting and ending points are fixed, but we can imagine that there are many paths that can be taken between these fixed points as shown in Figure 10.1.1 in the text and my Figure 1.

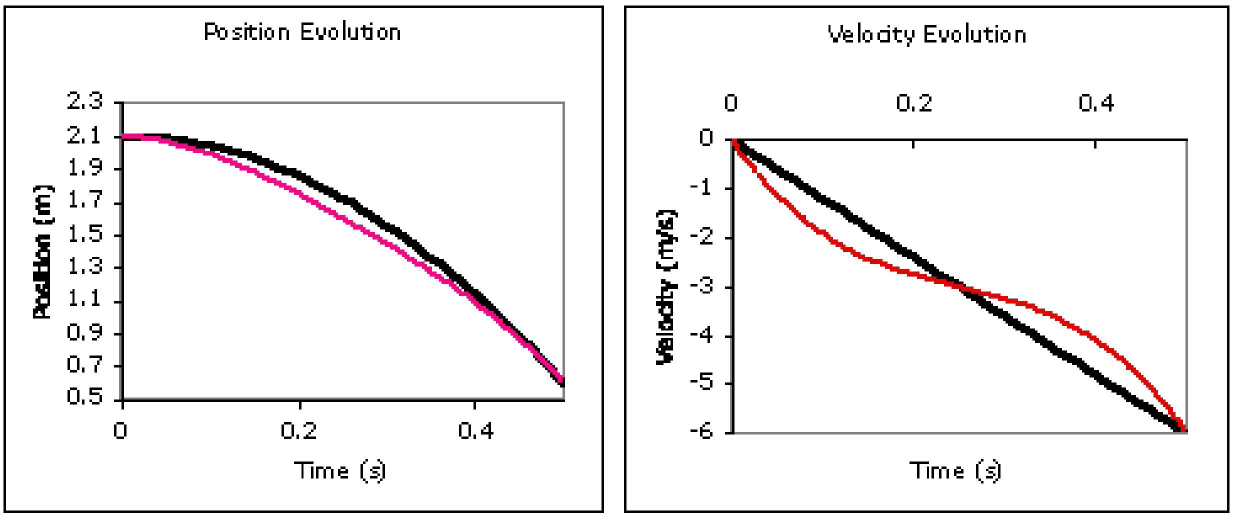

Figure 1: The true path and one possible variation from it. Note that the starting and ending positions, and the starting and ending velocities are identical for the two paths. The heavy line is the true path, parabolic motion $y = 2.1 - 6t^2$. The variation follows the equation $y = 2.1 + Bt^2 + Ct^3 + Dt^4$. D can be chosen arbitarily and the boundary conditions allow us to get the other coefficients. As shown, $B = -13.5, C = 30, D = -30$ in appropriate SI units.

Hamilton's Variational Principle states that the integral of the Lagrangian along the actual path of motion is an extremum (maximum or minimum) compared with nearby variational paths. In symbols we evaluate the integral

$$J = \int_{t_1}^{t_2} L\, dt \tag{2}$$

and say that

$$\delta J = \delta \int_{t_1}^{t_2} L\, dt = 0 \tag{3}$$

We must carefully define what we mean by δ. Recall that if we have a function of a **variable**, $F(x)$, its extrema are found by $dF/dx = 0$.

We are expressing something different in Equation 3: L, the integrand, is a function of two **parameters**, $y, \dot{y}$ (position and velocity) represented in 1D by paths on the $y-t,\ \dot{y}-t$ graphs. A change in the parameter, i.e. a change in the path taken between fixed start and end points, will change the result of the integral, and this integral is an extremum (maximum or minimum) for the actual path traveled.

In Figure 1, the true path is $y = 2.1 - 6t^2$ and the variation is $y = 2.1 + Bt^2 + Ct^3 + Dt^4$. B and C are found from the values at $t = 0$ and $t = 0.5$. If $D = 0$, then the boundary conditions are met for $C = 0$ and $B = -6$, i.e. the true solution. If $D = -30$ then $C = 30$ and $B = -13.5$. You can verify that at $t = 0$ and $t = 0.5$ both the true path and the variation have the same positions and velocities.

E.g. Consider the case of an object falling near the earth in 1D motion. We will show first that Hamilton's Principle leads to the usual differential equation. We have kinetic energy $T = m\dot{y}^2/2$ and potential energy $V = mgy$ so

$$L = \frac{1}{2}m\dot{y}^2 - mgy \tag{4}$$

To evaluate the variation we consider $\delta y, \delta\dot{y}$ to be slight variations in position and time from the extremum value. Thus

$$\delta J = \delta \int_{t_1}^{t_2} \left[\frac{1}{2}m\dot{y}^2 - mgy\right] dt = \int_{t_1}^{t_2} \left[m\,\dot{y}\,\delta\dot{y} - mg\,\delta y\right] dt \tag{5}$$

where this equation illustrates how to bring the variation inside the integral.

Now

$$\delta\dot{y} = \frac{d(\delta y)}{dt} \tag{6}$$

We evaluate the first term using integration by parts as

$$\int_{t_1}^{t_2} m\,\dot{y}\,\delta\dot{y}\,dt = m\dot{y}\,\delta y\Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} m\ddot{y}\,\delta y\,dt \tag{7}$$

Since the endpoints are fixed, with no variation, the first term on the right must be zero. Hence

$$\delta J = \int_{t_1}^{t_2} \left[-m\ddot{y} - mg\right] \delta y\,dt = 0 \tag{8}$$

Since δy is arbitrary, this variation can be zero only if

$$-m\ddot{y} - mg = 0 \tag{9}$$

E.g Come again? How does the δy stuff work? We consider the solution $y(\alpha, t)$ to be a function of the time t and a parameter α.

If we have initial conditions of $t = 0, y = 0, \dot{y} = 0$ the solution to the differential equation is $y(t) \equiv y(0,t) = -gt^2/2, \dot{y} \equiv \dot{y}(0,t) = -gt$. Now we write the variation from this solution as

$$y(\alpha, t) = y(0,t) + \alpha\eta(t) \tag{10}$$

Here $\eta(t)$ represents the variation, and is any function of time that (a) has a continuous first derivative (velocity) on the time interval $[t_1, t_2]$, and (b) has values $\eta(t_1) = 0 = \eta(t_2)$ so that the starting and ending points are fixed.

The parameter α is the strength of the variation. When $\alpha = 0$ we have our true solution.

From Equation 10 we have

$$\dot{y}(\alpha, t) = \dot{y}(0,t) + \alpha\dot{\eta}(t) \tag{11}$$

and to ensure that velocities match at the end points, $\dot{\eta}(t_1) = 0 = \dot{\eta}(t_2)$

Using these expressions we can get the kinetic and potential energies and hence the Lagrangian.

$$T = \frac{1}{2}m\left[\dot{y}(0,t) + \alpha\dot{\eta}(t)\right]^2 = \frac{1}{2}m\left[-gt + \alpha\dot{\eta}(t)\right]^2 = \frac{1}{2}mg^2t^2 - mg\,t\,\alpha\,\dot{\eta}(t) + \frac{1}{2}m\alpha^2\dot{\eta}^2(t) \tag{12}$$

$$V = mg\left[y(0,t) + \alpha\eta(t)\right] = -\frac{1}{2}mg^2t^2 + mg\,\alpha\,\eta(t) \tag{13}$$

$$L = m\left\{g^2t^2 - \alpha g\left[t\dot{\eta}(t) + \eta(t)\right] + \frac{1}{2}\alpha^2\dot{\eta}^2(t)\right\} \tag{14}$$

Now evaluate the integral of the Lagrangian. First evaluate the term linear in α, integrating by parts.

$$\alpha g\int_{t_1}^{t_2}\left[t\dot{\eta}(t) + \eta(t)\right]dt = \alpha g t\eta(t)\Big|_{t_1}^{t_2} - \alpha g\int_{t_1}^{t_2}\eta(t)dt + \alpha g\int_{t_1}^{t_2}\eta(t)dt = 0 \tag{15}$$

This means that

$$J(\alpha) = \int_{t_1}^{t_2} L\,dt = \frac{1}{3}mg^2(t_2^3 - t_1^3) + \frac{1}{2}m\alpha^2\int_{t_1}^{t_2}\dot{\eta}^2(t)dt \tag{16}$$

Now for any real function $\eta(t)$ the latter integral must be positive, so $J(\alpha)$ must be parabolic in α, and concave up. The minimum in this function occurs for $\alpha = 0$ as expected from Hamilton's Principle.

E.g. Motion of a particle in force-free region. We have Newton's First Law that says the motion should be a straight line. We will use Hamilton's Principle to show that if we assume that the motion is sinusoidal, the sinusoid must have zero amplitude, and hence be straight line motion between the start and end.

We suppose the particle to move from $t = 0, x = 0, y = 0$ to $t = x_1/v_x, x = x_1, y = 0$. The components of the position during motion are

$$x = v_x t \tag{17}$$
$$y = \pm\eta \sin \frac{\pi v_x t}{x_1} \tag{18}$$

where η, the amplitude of the sinusoid, can be varied. (Like α in the previous example.)

The potential energy is constant (but not necessarily zero), so

$$L = \frac{1}{2}m\left[v_x^2 + \left(\frac{\eta\pi v_x}{x_1}\right)^2 \cos^2 \frac{\pi v_x t}{x_1}\right] - V \tag{19}$$

Evaluating $J(\eta)$,

$$J(\eta) = \frac{m v_x x_1}{2} - V\frac{x_1}{v_x} + \eta^2 \frac{m v_x \pi^2}{4x_1} \tag{20}$$

and this is a minimum for $\eta = 0$.

2 Generalized Coordinates, Degrees of Freedom, Holonomic and Non-Holonomic Constraints

Next in the development is a discussion of *generalized coordinates.*

Recall Chapter 1 when we defined various 3D orthogonal coordinate systems, cartesian (x, y, z), cylindrical polar (r, θ, z) and spherical polar (r, θ, ϕ). These are all orthogonal meaning that $\hat{e}_a \cdot \hat{e}_b = 0$ where a and b are two of the unit vectors for a particular coordinate system.

Generalized coordinates by contrast do not need to be orthogonal, but are chosen to best represent a complicated system.

E.g. A pendulum constrained to move in the $x-y$ plane. Initially we might choose cartesian coordinates, however they do not encapsulate two factors: the plane of oscillation is $x - y$ and the length of the pendulum, r, is fixed: i.e. $z = 0$ and $r^2 - x^2 - y^2 = 0$. It is more convenient to choose cylindrical polar coordinates with $z = 0$, i.e. plane polar coordinates. The second constraint is $r = constant$ meaning that there is only one variable, θ, and we can describe the motion as "one dimensional" meaning a single variable describes the motion. Similarly motion of a roller coaster car is one dimensional.

Generalized coordinates q_i are a set of coordinates that are independent, and just sufficient to uniquely specify the configuration of a system.

Degrees of freedom are the number of variables that are needed to describe a system. Thus a single particle in 3D space has 3 degrees of freedom (x, y, z) while two particles in 3D space have 6 degrees of freedom, $(x_1, y_1, z_1, x_2, y_2, z_2)$. However a rigid dumbbell composed of two particles has only 5 degrees of freedom. A good choice of generalized coordinates for the dumbbell would be (x, y, z) for the center of mass plus (θ, ϕ) for the orientation of the rod joining the particles (see Figure 10.2.2).

Conditions of constraint are statements about limitations on motion of a system. They are categorized into *holonomic* and *non-holonomic* types.

Holonomic constraints can be represented by an equality involving coordinates such as $d^2 - \left[(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2\right] = 0$.

Being on a surface, or having a fixed separation between two particles are holonomic constraints.

Non-holonomic constraints include those that involve an inequality (being outside the earth) or involving velocity constraints. We will focus on holonomic constraints and leave non-holonomic constraints for an advanced mechanics course.

If we have N particles and m holonomic constraints, there are $(3N - m)$ degrees of freedom and the same number of generalized coordinates. Thus for the dumbbell, $N = 2, m = 1$ and there are $3(2) - 1 = 5$ degrees of freedom, 5 generalized variables.

3 Many Paths to One Answer: Getting T and V in terms of generalized coordinates

Here we will look at a more complicated system (one we would NOT want to tackle with just Newton's Laws) and get the same result three different ways.

The system consists of two particles. One has mass M and is constrained to move in a straight line on a frictionless surface. Attached to it is a simple pendulum of length r with mass m. We want to find kinetic and potential energies so that we can construct the Lagrangian.

We have $N = 2$ particles, and $m = 4$ constraints ($Y = Z = 0$ for the sliding block, $z = 0, r = constant$ for the pendulum) so there must be $(3(2) - 4) = 2$ generalized coordinates. The natural choice for generalized coordinates are the position of the first mass, X, and the angle that the pendulum makes with the vertical, θ. Refer to Figure 10.3.1 in the text and my Figure 2. In Cartesian coordinates we have the position of the block, X and the position of the ball x, y.

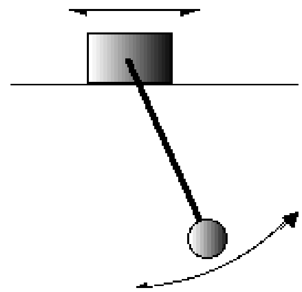

Figure 2: Block slides on a horizontal surface with no friction. Simple pendulum swings with no friction at axis.

First Path to Lagrangian Start with the cartesian coordinates, (X, Y, Z) and (x, y, z), measured from a single inertial frame. The 4 constraints can be written $Y = Z = z = 0$ and $(x - X)^2 + y^2 - r^2 = 0$. This means we have two generalized coordinates (duh, we just said that!)

In terms of Cartesian coordinates,

$$\begin{aligned} T &= \frac{1}{2}M\dot{X}^2 + \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) \\ V &= mgy \end{aligned} \tag{21}$$

Now write transformation equations from the Cartesian coordinates to the generalized coordinates.

$$x = X + r\sin\theta \tag{22}$$

$$y = -r\cos\theta \tag{23}$$

$$X = X \tag{24}$$

Hence

$$\dot{x} = \dot{X} + r\dot{\theta}\cos\theta \tag{25}$$

$$\dot{y} = r\dot{\theta}\sin\theta \tag{26}$$

$$\dot{X} = \dot{X} \tag{27}$$

and using these in Equations 21

$$T = \frac{1}{2}M\dot{X}^2 + \frac{1}{2}m\left[\dot{X}^2 + (r\dot{\theta})^2 + 2\dot{X}r\dot{\theta}\cos\theta\right] \tag{28}$$

$$V = -mgr\cos\theta \tag{29}$$

$$L = \frac{1}{2}M\dot{X}^2 + \frac{1}{2}m\left[\dot{X}^2 + (r\dot{\theta})^2 + 2\dot{X}r\dot{\theta}\cos\theta\right] + mgr\cos\theta \tag{30}$$

Notice that

- The kinetic energy term has a mixed term in it. The generalized coordinates are not orthogonal.
- The potential energy depends on a single generalized coordinate (this may not be true in general.)
- The Lagrangian has no terms in X. In cases where this is true we call X an *ignorable* variable.

Second Path to Lagrangian Using generalized coordinates at the start.

The velocity of M is $\vec{V}_M = \hat{i}\dot{X}$ and the velocity of m is the velocity of M plus the velocity of m relative to M.

$$\vec{v}_m = \vec{V}_M + \vec{v}_{m,rel} = \hat{i}\dot{X} + \hat{e}_\theta r\dot{\theta} \tag{31}$$

Then since $\hat{e}_\theta = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$, $\dot{x} = \dot{X} + r\dot{\theta}\cos\theta$ and $\dot{y} = r\dot{\theta}\sin\theta$ as in Equations 25, 26 in the First Path to the Lagrangian. The Lagrangian follows from that treatment.

Third Path to the Lagrangian Similar to the Second Path

If we write

$$T = \frac{1}{2}M\vec{V}_M \cdot \vec{V}_M + \frac{1}{2}m\vec{v}_m \cdot \vec{v}_m \tag{32}$$

then using $\hat{i} \cdot \hat{e}_\theta = \cos\theta$ we find the expression for kinetic energy in Equation 28.

Usually either the First Path or the Third Path will be helpful.

4 Lagrange's Equations of Motion

Now we apply Hamilton's Variational Principle to the situation of a Lagrangian as a function of generalized coordinates (q_i, $\dot{q}_i$, $i = 1 \cdots N$) and generate the Lagrangian Equations of motion.

$$\delta J = \delta \int_{t_1}^{t_2} L\,dt = \int_{t_1}^{t_2} \delta L\,dt = \int_{t_1}^{t_2} \sum_i \left(\frac{\partial L}{\partial q_i}\,\delta q_i + \frac{\partial L}{\partial \dot{q}_i}\,\delta \dot{q}_i \right) dt = 0 \tag{33}$$

The q_i are parameters that are functions of time, and we can write

$$\delta \dot{q}_i = \frac{d(\delta q_i)}{dt} \tag{34}$$

We can use this to evaluate the second term of Equation 33 using integration by parts.

$$\int_{t_1}^{t_2} \sum_i \left(\frac{\partial L}{\partial \dot{q}_i}\,\delta \dot{q}_i \right) dt = \sum_i \frac{\partial L}{\partial \dot{q}_i}\delta q_i \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \sum_i \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) \delta q_i\,dt \tag{35}$$

The first term in Equation 35 vanishes at the endpoints ($\delta q_1 = \delta q_2 = 0$) and so

$$\delta \int_{t_1}^{t_2} L\, dt = \int_{t_1}^{t_2} \sum_i \left[\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\right] \delta q_i\, dt = 0 \tag{36}$$

The generalized coordinates are independent, and so each δq_i is independent of the other variations. Thus to ensure that the integral is identically zero for all variations the item in square brackets must be zero for each generalized coordinate. The Lagrange equations of motion are thus

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0 \qquad i = 1, 2, \cdots, N \tag{37}$$

This can also be written as

$$\frac{\partial L}{\partial q_i} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) \qquad i = 1, 2, \cdots, N \tag{38}$$

In Section 6 we will define a conjugate generalized momentum for each generalized coordinate, $p_i = \frac{\partial L}{\partial \dot{q}_i}$ and the equations of motion can be written

$$\dot{p}_i \equiv \frac{dp_i}{dt} = \frac{\partial L}{\partial q_i} \tag{39}$$

Equations 37, 38, and 39 are all equivalent. Now we need to see how to use them.

5 Some Applications of the Lagrange Formulation

The general strategy consists of

1. Select a suitable set of generalized coordinates
2. Find transformation equations between the cartesian and generalized coordinates
3. Write the kinetic energy as a function of generalized coordinates using the ideas from Section 3, Path 1 or Path 3
4. Write the potential energy as a function of generalized coordinates
5. Construct the Lagrangian and do derivatives to get the equations of motion (use one of Equations 37, 38, and 39.)

The result is a set of equations of motion that must be solved by the methods of differential equations, some of which we have discussed earlier in the text.

5.1 Using Lagragian to get things we already know

E.g. 1 Free particle Yup this is boring, but let's make sure it works for a particle moving in 1D with no forces, therefore no potential energy. There is only one generalized coordinate, x. The Lagrangian is simply $L = T = \frac{1}{2}m\dot{x}^2$.

Then the partial derivatives are $\frac{\partial L}{\partial x_i} = 0$ and $\frac{\partial L}{\partial \dot{x}_i} = m\dot{x}$ and the equation of motion is

$$0 = \frac{d}{dt}(m\dot{x}) \tag{40}$$

i.e. linear momentum is constant, just a restatement of Newton's First Law.

E.g. 2 Projectile Motion in Uniform g Imagine motion in the $x - z$ plane. The generalized coordinates are just (x, z) and we can quickly write the Lagrangian

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{z}^2) - mgz \tag{41}$$

The partials are

$$\frac{\partial L}{\partial x} = 0 \qquad \frac{\partial L}{\partial z} = -mg \qquad \frac{\partial L}{\partial \dot{x}_i} = m\dot{x} \qquad \frac{\partial L}{\partial \dot{z}_i} = m\dot{z} \tag{42}$$

so the equations of motion are

$$0 = \frac{d}{dt}(m\dot{x}) \quad \text{i.e. horizontal momentum is constant} \tag{43}$$

$$-mg = \frac{d}{dt}(m\dot{z}) = m\ddot{z} \tag{44}$$

and these are just what we get from Newton's Laws

E.g. 3 Harmonic Oscillator in 1D We have a single generalized coordinate x representing the displacement of the mass from the unstretched location of the spring.

The Lagrangian is $L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$ and the partials are

$$\frac{\partial L}{\partial x} = -kx \qquad \frac{\partial L}{\partial \dot{x}} = m\dot{x} \tag{45}$$

and our equation of motion is simply

$$-kx = m\ddot{x} \tag{46}$$

which is just our usual differential equation.

E.g. 4 Central Force in 2D Here we have a constraint, $z = 0$ so we need 3(1)-1 = 2 generalized coordinates. We choose the polar coordinates (r, θ) and have transformation equations (First Path to Lagrangian)

$$x = r\cos\theta \qquad y = r\sin\theta \qquad \dot{x} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta \qquad \dot{y} = \dot{r}\sin\theta + r\dot{\theta}\cos\theta \tag{47}$$

Since the potential energy comes from a central force, it only depends on r so we write the Lagrangian

$$L = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - V(r) \tag{48}$$

The text illustrates getting the Lagrangian using the Third Path.

The partials are

$$\frac{\partial L}{\partial \dot{r}} = m\dot{r} \qquad \frac{\partial L}{\partial r} = mr\dot{\theta}^2 - \frac{\partial V}{\partial r} \qquad \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} \qquad \frac{\partial L}{\partial \theta} = 0 \tag{49}$$

Recognizing that the force is $f(r) = -\partial V/\partial r$, we write the equations of motion as

$$m\ddot{r} = mr\dot{\theta}^2 + f(r) \qquad \frac{d}{dt}(mr^2\dot{\theta}) = 0 \tag{50}$$

These are familiar from Chapter 6.

Notice that anytime a generalized coordinate (like θ for central forces) does not appear in the Lagrangian, then the conjugate momentum (yet to be discussed), $\partial L/\partial \dot{q}_i$ is constant. In this example the angular momentum, $mr^2\dot{\theta}$ is constant.

5.2 More difficult problems

E.g. 5 The Complete Atwood's Machine Consider a pulley mounted on a fixed, frictionless bearing having moment of inertia I and radius a. A string of length ℓ connects two blocks of masses m_1, m_2 that can only move vertically. We neglect any air resistance and assume an ideal string (massless and non-stretchy) that does not slip on the pulley. This is shown in Figure 10.5.1 and my Figure 3.

There are 5 holonomic constraints, $y_1 = z_1 = y_2 = z_2 = 0$ due to planar motion in one direction, and $\ell =$ constant for the string length.

We use x for the position of m_1 and then the position of m_2 is $(\ell - \pi a - x)$. We also know that since the string does not slip, $\omega = \dot{x}/a$.

The Lagrangian is then

$$L = \frac{1}{2}\left(m_1 + m_2 + \frac{I}{a^2}\right)\dot{x}^2 + m_1 gx + m_2 g(\ell - \pi a - x) \tag{51}$$

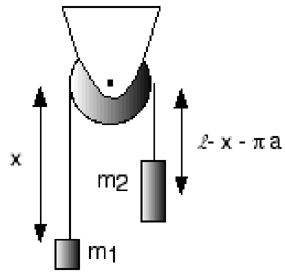

Figure 3: Atwood's Machine. The pulley is of mass m and radius a, and the string does not slip on the pulley. There is no friction at the axle.

The partials are

$$\frac{\partial L}{\partial x} = (m_1 - m_2)g \qquad \frac{\partial L}{\partial \dot{x}} = \left(m_1 + m_2 + \frac{I}{a^2} \right) \dot{x} \tag{52}$$

Leading to the equation of motion

$$\left(m_1 + m_2 + \frac{I}{a^2} \right) \ddot{x} = (m_1 - m_2)g \tag{53}$$

meaning that the acceleration is constant with value

$$\frac{(m_1 - m_2)g}{m_1 + m_2 + \frac{I}{a^2}} \tag{54}$$

This should be familiar from Physics 312.

E. g. 6 The Double Atwood Machine Replace the second object by a second Atwood's machine with masses m_2, m_3. Simplify the problem by making the two moments of inertia zero, and the pulley radii negligible. The two string lengths will be ℓ and ℓ'. See Figure 10.5.2 in text or my Figure 4.

The generalized coordinates will be x, the position of m_1 relative to the first pulley, and x', the position of m_2 relative to the moving pulley.

The holonomic constraints are for the three masses and the moving pulley, with no motion in the y, z directions (8 constraints) plus constraints on the string lengths (2 constraints.) Verifying, we need (3(4) - 8 - 2 = 2) generalized coordinates.

The First Path to the Lagrangian is easy.

$$x_1 = x \qquad x_p = \ell - x \qquad x_2 = x_p + x' = \ell - x + x' \qquad x_3 = x_p + (\ell' - x') = \ell + \ell' - x - x' \tag{55}$$

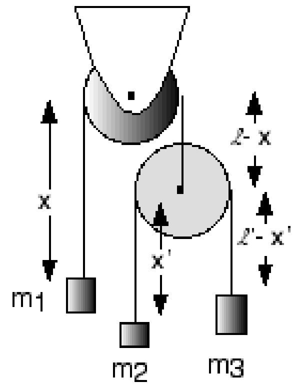

Figure 4: A double Atwood's machine. Assume that the pulleys have no mass and negligible radius.

Here we have defined down as positive x, so the potential energy is $-mgx$.

Hence we get the Lagrangian

$$L = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2(-\dot{x}+\dot{x}')^2 + \frac{1}{2}m_3(\dot{x}+\dot{x}')^2 + (m_1-m_2-m_3)gx + (m_2-m_3)gx' + Constants \tag{56}$$

The partials are

$$\frac{\partial L}{\partial x} = (m_1 - m_2 - m_3)g \qquad \frac{\partial L}{\partial \dot{x}} = m_1\dot{x} - m_2(-\dot{x}+\dot{x}') + m_3(\dot{x}+\dot{x}')$$

$$\frac{\partial L}{\partial x'} = (m_2 - m_3)g \qquad \frac{\partial L}{\partial \dot{x}'} = m_2(-\dot{x}+\dot{x}') + m_3(\dot{x}+\dot{x}') \tag{57}$$

and the equations of motion are

$$(m_1 + m_2 + m_3)\ddot{x} + (-m_2 + m_3)\ddot{x}' = (m_1 - m_2 - m_3)g \tag{58}$$

$$(-m_2 + m_3)\ddot{x} + (m_2 + m_3)\ddot{x}' = (m_2 - m_3)g \tag{59}$$

These simultaneous equations can be easily solved, although the reult is messy with symbols. Try it with numbers. Let $m_1 = 5$ kg, $m_2 = 2$ kg, and $m_3 = 3$ kg. Find the accelerations $\ddot{x}$ and $\ddot{x}'$and then find the acceleration of each mass. Next try reversing the values for m_2, m_3.

E. g. 7 Block Sliding on a Frictionless Wedge that is Free to Move on Frictionless Table A wedge of mass M and angle θ can slide freely in 1D on a horizontal frictionless table. A box of mass m slides along the frictionless incline of the wedge. Find the equations of motion from the Lagrangian.

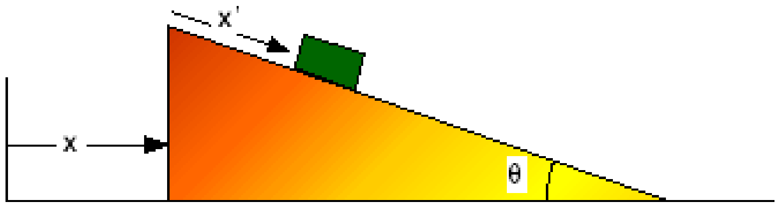

Figure 5: The wedge has mass M and the block has mass m. There is no friction.

As suggested in Figure 10.5.3 we will have two generalized coordinates, x for the horizontal location of the wedge and x' for the position of the box relative to the top of the wedge. We will use Path 3 to get the kinetic energy.

The velocity of the wedge is $\vec{V} = \dot{x}\hat{i}$ and the velocity of the box is $\vec{v} = \dot{x}\hat{i} + \dot{x}'\hat{i}'$.

The kinetic energies are

$$T_M = \frac{1}{2}M\vec{V}\cdot\vec{V} = \frac{1}{2}M\dot{x}^2 \qquad T_m = \frac{1}{2}m\vec{v}\cdot\vec{v} = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{x}'^2 + m\dot{x}\dot{x}'\cos\theta \tag{60}$$

The potential energy is $V = -mg\,x'\sin\theta$ and the Lagrangian is

$$L = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{x}'^2 + m\dot{x}\dot{x}'\cos\theta + mg\,x'\sin\theta \tag{61}$$

with partials

$$\frac{\partial L}{\partial x} = 0 \qquad \frac{\partial L}{\partial \dot{x}} = (M+m)\dot{x} + m\dot{x}'\cos\theta \tag{62}$$

$$\frac{\partial L}{\partial x'} = mg\sin\theta \qquad \frac{\partial L}{\partial \dot{x}'} = m\dot{x}' + m\dot{x}\cos\theta \tag{63}$$

With the first partial being zero, this means that $(M+m)\dot{x} + m\dot{x}'\cos\theta =$ Constant. In a moment we will refer to this a a generalized momentum conjugate to x being conserved. For this problem this means that the horizontal net momentum is constant.

The two equations of motion are

$$(m+M)\ddot{x} + (m\cos\theta)\ddot{x}' = 0 \qquad \ddot{x}' + \ddot{x}\cos\theta = g\sin\theta \tag{64}$$

and we can solve these simultaneous equations to yield

$$\ddot{x} = \frac{-g\sin\theta\cos\theta}{(1+M/m) - \cos^2\theta} \qquad \ddot{x}' = \frac{g\sin\theta}{1 - m\cos^2\theta/(m+M)} \tag{65}$$

Try to think of all limiting cases to check the validity of these equations.

6 Generalized (Conjugate) Momenta and Ignorable Coordinates

It is convenient to define a *generalized momentum* to match with each generalized coordinate. We define the *generalized momentum conjugate to the generalized variable q* (conjugate momentum for short) as

$$p_q = \frac{\partial L}{\partial \dot{q}} \tag{66}$$

and thus can write the Lagrange equation of motion as

$$\dot{p}_q \equiv \frac{dp_q}{dt} = \frac{\partial L}{\partial q} \tag{67}$$

We have done several examples of Lagrangian analysis. For projectile motion $L = \frac{1}{2}m(\dot{x}^2 + \dot{z}^2) - mgz$ so the conjugate momenta are $p_x = m\dot{x}$ and $p_z = m\dot{z}$, both with units of linear momentum.

For central forces we found $L = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - V(r)$ so that the conjugate momenta are $p_r = m\dot{r}$ and $p_\theta = mr^2\dot{\theta}$, with one having the units of linear momentum and the other the units of angular momentum.

In the above two examples the conjugate momentum is clearly one of the momenta that we would write down in the Newtonian formulation. In many cases the conjugate momentum is more complicated. Consider the Atwood's Machine with the Lagrangian of Equation 51. The conjugate momentum is $p_x = \left(m_1 + m_2 + I/a^2\right)\dot{x}$, but even here we might recognize this as the "system momentum".

For the block sliding on the wedge with Lagrangian given by Equation 61, the momenta are less obvious,

$$p_x = (M+m)\dot{x} + m\dot{x}'\cos\theta \qquad p_{x'} = m\dot{x}' + m\dot{x}\cos\theta$$

In several examples, such as the central force problem, the Lagrangian did not include one of the generalized coordinates. For the central force problem θ does not appear in the Lagrangian. In these cases the text refers to the coordinate as "*ignorable*".

For an ignorable coordinate q, $\partial L/\partial q = 0$ and therefore $p_q = \partial L/\partial \dot{q}$ is constant—Conservation of Momentum. Generalized coordinates and their conjugate momenta become central in our definition of the Hamiltonian at the end of this chapter.

E.g. 8 Pendulum Attached to Movable Support In Section 3 we began discussing a mass M constrained to move in a straight line on a frictionless surface with a simple pendulum of length r, mass m attached to it. The generalized coordinates were X and θ and the Lagrangian was

$$L = \frac{1}{2}M\dot{X}^2 + \frac{1}{2}m\left[\dot{X}^2 + (r\dot{\theta})^2 + 2\dot{X}r\dot{\theta}\cos\theta\right] + mgr\cos\theta \tag{68}$$

X is ignorable so

$$p_X = (M+m)\dot{X} + mr\dot{\theta}\cos\theta = \text{Constant} \tag{69}$$

and the equations of motion are

$$(M+m)\ddot{X} + mr\ddot{\theta}\cos\theta - mr\dot{\theta}^2\sin\theta = 0 \tag{70}$$

$$\frac{d}{dt}\left(r^2\dot{\theta} + \dot{X}r\cos\theta\right) = -(\dot{X}r\dot{\theta} + gr)\sin\theta$$

or

$$\ddot{\theta} + \frac{\ddot{X}}{r}\cos\theta + \frac{g}{r}\sin\theta = 0 \tag{71}$$

The last two equations can be combined to eliminate $\ddot{X}$ and yield a rather ugly looking second order differential equation in θ.

The text makes some simplifying assumptions to see if these equations make sense. Read and understand these

E.g. 9 Spherical Pendulum Including Approximations Return to the simple pendulum, a mass m attached to a string of length ℓ that s fixed at the other end, but now let the pendulum be free to move in 3D.

This problem is identical to that of a spherical bowl of radius ℓ with a frictionless bar of soap free to move in it.

There are two degrees of freedom, (θ, ϕ), the usual spherical polar variables. The velocity (Equation 1.12.12) is

$$\vec{v} = \hat{e}_\theta r\dot{\theta} + \hat{e}_\phi r\sin\theta\,\dot{\phi} \tag{72}$$

If we choose the lowest position for $V = 0$, the height is $y = \ell(1-\cos\theta)$ and the Lagrangian is

$$L = \frac{1}{2}m\ell^2(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta) - mg\ell(1-\cos\theta) \tag{73}$$

Since ϕ is ignorable we have conservation of the conjugate momentum,

$$p_\phi = m\ell^2\dot{\phi}\sin^2\theta \equiv m\ell^2 S \tag{74}$$

where we have defined a new constant $S = \dot{\phi}\sin^2\theta$.

The remaining equation of motion is then

$$\frac{d}{dt}\left(m\ell^2\dot{\theta}\right) = m\ell^2\ddot{\theta} = m\ell^2\dot{\phi}^2 \sin\theta\cos\theta - mg\ell\sin\theta$$

or

$$\ddot{\theta} + \frac{g}{\ell}\sin\theta - S^2\frac{\cos\theta}{\sin^3\theta} = 0 \tag{75}$$

We shall NOT try to solve this in general, but will look at three special cases: (a) simple plane pendulum, $\dot{\phi} = 0$, (b) Conical pendulum, $\dot{\theta} = 0$, and (c) slight perturbation from a conical pendulum.

1. For the simple plane pendulum, $\dot{\phi} = 0$, we have $S = 0$ and the equation of motion becomes

$$\ddot{\theta} + \frac{g}{\ell}\sin\theta = 0 \tag{76}$$

which is just what we have seen before for the plane pendulum. So we gain confidence that our derivation is correct.

2. For a conical pendulum, $\dot{\theta} = 0$, the bob moves in a circle with the string having a constant angle θ_0 and the equation of motion becomes

$$\frac{g}{\ell}\sin\theta - S^2\frac{\cos\theta}{\sin^3\theta} = 0$$

or, putting in the expression for S

$$\dot{\phi}^2 = \frac{g}{\ell\cos\theta} = \frac{g}{\ell}\sec\theta \tag{77}$$

and this gives the required angular velocity $\dot{\phi}$ for the conical motion.

3. Now suppose we disturb the conical pendulum slightly from the above value. To indicate the perfect conical pendulum value we will add a subscript 0

$$\dot{\phi}_0^2 = \frac{g}{\ell\cos\theta_0} = \frac{g}{\ell}\sec\theta_0 \tag{78}$$

Also we will express the ideal conical values of S_0 and period T_0

$$S_0^2 = \frac{g}{\ell}\sin^4\theta_0\sec\theta_0 \tag{79}$$

$$T_0 = \frac{2\pi}{\dot{\phi}} = 2\pi\sqrt{\frac{\ell\cos\theta_0}{g}} \tag{80}$$

We assume that the perturbation will not change the value of S_0 significantly (nor will it change the period T_0) and write the equation of motion as

$$\ddot{\theta} + \frac{g}{\ell}\left(\sin\theta - \frac{\sin^4\theta_0}{\cos\theta_0}\frac{\cos\theta}{\sin^3\theta}\right) = 0 \tag{81}$$

Now consider the expression in parentheses

$$f(\theta) = \sin\theta - \frac{\sin^4\theta_0}{\cos\theta_0}\frac{\cos\theta}{\sin^3\theta} \tag{82}$$

Do a Taylor series expansion about $\theta = \theta_0$ to first order to get

$$f(\theta) \approx (3\cos\theta_0 + \sec\theta_0)(\theta - \theta_0) \tag{83}$$

Introduce the variable (pronounced *ksi*) $\xi = \theta - \theta_0$, with $\ddot{\xi} = \ddot{\theta}$ so that the equation of motion is

$$\ddot{\xi} + \frac{g}{\ell}(3\cos\theta_0 + \sec\theta_0)\xi \tag{84}$$

This is our familiar friend the harmonic oscillator with period

$$T_1 = 2\pi\sqrt{\frac{\ell}{g(3\cos\theta_0 + \sec\theta_0)}} \tag{85}$$

So as the pendulum makes its primarily circular motion with period T_0, it bobs up and down with period T_1.

E.g. Suppose that $\ell = 60$ cm and $\theta_0 = 35°$. Then $T_0 = 1.407$ s and $T_1 = 0.811$ s.

The maxima in θ are a good visual reference to the motion. In the time T_1 between successive maxima, the bob rotates through an angle

$$\phi \approx \dot{\phi}_0 T_1 = \frac{2\pi}{\sqrt{3\cos^2\theta_0 + 1}} > \pi \tag{86}$$

and so the point of maximum θ precesses in the direction of increasing ϕ as shown in Figure 10.6.2. The text integrates the equation of motion once to get an effective potential that it plots versus angle to show the angular turning points.

7 Forces of Constraint: Lagrange Multipliers

The previous sections have shown how the Lagrangian lets us determine constants of motion and equations of motion. We may also want to determine the size of the constraint forces such as normal forces or tensions. Rather than return to Newton's Methods, we introduce another new powerful method, Lagrange multipliers.

In the Lagrangian method used previously we used the constraints to reduce the number of variables needed to describe the system, and to make all the remaining variables independent.

7.1 Text Development

Consider a system with two generalized coordinates, (q_1, q_2) connected by a single equation of constraint, $f(q_1, q_2, t) = 0$ where a time dependent constraint has been allowed. In the Lagrangian approach already used we would use the constraint to reduce the system to a single generalized coordinate. Here we will NOT do this step so that we can determine a force of constraint.

Start with Hamilton's Variational Principle from Section 4,

$$\delta \int_{t_1}^{t_2} L\, dt = \int_{t_1}^{t_2} \sum_i \left[\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\right] \delta q_i\, dt = 0 \tag{87}$$

In Section 4 the constraints had already been used, so the δq's were all independent. In this section we are starting earlier and have not used the constraint, so the δq are NOT independent. We write the variation in the constraint, which at any time must be zero.

$$\delta f = \left(\frac{\partial f}{\partial q_1}\delta q_1 + \frac{\partial f}{\partial q_2}\delta q_2 \right) = 0 \tag{88}$$

Hence

$$\delta q_2 = -\left(\frac{\partial f/\partial q_1}{\partial f/\partial q_2}\right)\delta q_1 \tag{89}$$

Putting this into Equation 87,

$$\int_{t_1}^{t_2} \left[\left(\frac{\partial L}{\partial q_1} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_1}\right) - \left(\frac{\partial L}{\partial q_2} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_2}\right)\left(\frac{\partial f/\partial q_1}{\partial f/\partial q_2}\right)\right]\delta q_1\, dt = 0 \tag{90}$$

Now we are varying a single coordinate, and the variation can take any value. To make the integral zero, therefore, the item in square brackets must be zero, or

$$\frac{(\partial L/\partial q_1) - (d/dt)(\partial L/\partial \dot{q}_1)}{(\partial f/\partial q_1} = \frac{(\partial L/\partial q_2) - (d/dt)(\partial L/\partial \dot{q}_2)}{(\partial f/\partial q_2} \tag{91}$$

The text states that the left hand side is an expression that is a function of $(q_1, \dot{q}_1, t)$ with time being either implicitly (via $q_1(t), \dot{q}_1(t)$) or explicitly involved. Likewise the right hand side is a function of $(q_2, \dot{q}_2, t)$. For these to be equal for arbitrary choices of the generalized variables, they must both equal a function of time alone, that we call $-\lambda(t)$, a Lagrange multiplier.

Then we can write

$$\frac{\partial L}{\partial q_1} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_1} + \lambda(t)\frac{\partial f}{\partial q_1} = 0 \tag{92}$$

$$\frac{\partial L}{\partial q_2} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_2} + \lambda(t)\frac{\partial f}{\partial q_2} = 0 \tag{93}$$

These two equations plus the constraint condition provide three equations in the three unknowns $(q_1(t), q_2(t), \lambda(t))$ that can be solved.

The quantities

$$Q_i = \lambda(t)\frac{\partial f}{\partial q_i} \tag{94}$$

are called the generalized forces and are either a force for a spacelike q or a torque for an angular q.

The text generalizes to more than 2 generalized coordinates and more than one constraint.

7.2 Alternate Method from *Analytical Mechanics* By Louis N. Hand, Janet D. Finch, available as a Google Book

We will start with an example that is more straightforward calculus: Consider the function $F(x,y) = x^2+y^2$. What values of (x,y) will minimize this function subject to the constraint $y = 2x+1$.

Geometrically we can see that the constraint describes a straight line on the $x-y$ plane, and F is the square of the distance from the origin to the line.

Standard calculus would use the constraint equation to eliminate y in F, then $dF/dx = 0$ could be solved for the desired value, $x = -0.4$. Then $y = 0.2$.

Instead we introduce a Lagrange multiplier λ and write $F' = x^2 + y^2 + \lambda(y - 2x - 1)$ and get three equations, $\partial F'/\partial x = 0, \quad \partial F'/\partial y = 0 \quad \partial F'/\partial \lambda = 0$ or in this case

$$\begin{aligned} 2x - 2\lambda &= 0 & (95)\\ 2y + \lambda &= 0 & (96)\\ y - 2x - 1 &= 0 & (97) \end{aligned}$$

Solving the simultaneous equations we get $x = \lambda = -0.4, y = 0.2$.

Try this one: Let $F(x) = 3x^2 + 2y^2$ and the constraint be $x = \cos(2y)$. Use Lagrange multipliers to minimize F and show that the answers are $x = 0.2293, y = 0.6697, \lambda = 1.3761$.

Returning to the mechanics realm, and for the case of a single constraint equation $f(q_1, q_2, t) = 0$, we introduce a modified Lagrangian,

$$L' = L + \lambda f \tag{98}$$

Then the equation of motion becomes

$$\frac{\partial L'}{\partial q_i} - \frac{d}{dt}\frac{\partial L'}{\partial \dot{q}_i} = 0 \tag{99}$$

7.3 Examples

E.g. 1 The Yo-Yo: a disk rolling down a string. Consider a disk of radius a, mass m and moment of inertia $I_{cm} = mk_{cm}^2$ that has a string wrapped around it. One end of the string is fixed and the yo-yo rolls down the string without slipping. We will assume that the string remains vertical.

After some time the yo-yo will haven fallen a distance y and rotated through an angle ϕ. Down is chosen as positive for y and clockwise is positive for ϕ. The constraint equation is

$$f(y,\phi) = y - a\phi = 0 \tag{100}$$

First let us apply the Lagrangian method incorrectly. The Lagrangian in terms of y and ϕ is

$$L_{incorrect} = \frac{1}{2}m\dot{y}^2 + \frac{1}{2}mk_{cm}^2\dot{\phi}^2 + mgy.$$

Writing the resulting equations of motion we get

$$\ddot{y} = g \tag{101}$$

$$mk_{cm}^2\dot{\phi}^2 = constant \tag{102}$$

This is clearly wrong—the yo-yo is not in free fall, and the angular velocity is not constant. Can you see where I made a mistake?

Now use the method of Lagrange multipliers. The modified Lagrangian is

$$L' = \frac{1}{2}m\dot{y}^2 + \frac{1}{2}mk_{cm}^2\dot{\phi}^2 + mgy + \lambda(y - a\phi) \tag{103}$$

Notice that ϕ is ignorable.The generalized momenta are

$$p_y = m\dot{y} \qquad p_\phi = mk_{cm}^2\dot{\phi} \tag{104}$$

Putting these into Equation 92, and writing the second time derivative of the constraint equation we get

$$mg + \lambda - m\ddot{y} = 0 \tag{105}$$

$$-\lambda a - mk_{cm}^2\ddot{\phi} = 0 \tag{106}$$

$$\ddot{y} - a\ddot{\phi} = 0 \tag{107}$$

These can be solved algebraically resulting in

$$\lambda = -\frac{mgk_{cm}^2/a^2}{1+k_{cm}^2/a^2} \tag{108}$$

$$\ddot{y} = \frac{1}{1+k_{cm}^2/a^2}\, g \tag{109}$$

$$\ddot{\phi} = \frac{1}{1+k_{cm}^2/a^2}\, \frac{g}{a} \tag{110}$$

Checking the limiting case of $k_{cm} = 0$ we get $\ddot{y} = g$ as expected. For a disk (done in the text) $k_{cm} = a^2/2$ and $\ddot{y} = \frac{2}{3}g$.

To find the tension in the string, use Equation 94 with the variable y.

$$T = \lambda\frac{\partial f}{\partial y} = -\frac{k_{cm}^2/a^2}{1+k_{cm}^2/a^2}\, mg \tag{111}$$

The generalized force related to ϕ is the torque of the tension relative to the center of mass.

E.g. 2 Ball rolling down a ramp A ball of radius R and moment of inertia $I_{cm} = mk_{cm}^2$ rolls without slipping down a slope inclined at α to the horizontal. Find the equations of motion and the generalized forces.

Choose up the incline to be positive x and positive θ to be consistent with this. The constraint between these variables is $x = R\theta$.

The modified Lagrangian is then

$$L' = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}I_{cm}\dot{\theta}^2 - mgx\sin\alpha + \lambda(x - R\theta) \tag{112}$$

We have two equations of motion that come from this modified Lagrangian, plus the constraint equation,

$$-mg\sin\theta + \lambda - m\ddot{x} = 0 \tag{113}$$

$$-\lambda R - I_{cm}\ddot{\theta} = 0 \tag{114}$$

$$x - R\theta = 0 \tag{115}$$

These are solved in a trivial fashion to get (using the radius of gyration k_{cm})

$$\ddot{x} = -g\sin\alpha\left(\frac{R^2}{R^2+k_{cm}^2}\right) \tag{116}$$

$$\ddot{\theta} = -g\sin\alpha\left(\frac{R}{R^2+k_{cm}^2}\right) \tag{117}$$

$$\lambda = mg\sin\alpha\left(\frac{k_{cm}^2}{R^2+k_{cm}^2}\right) \tag{118}$$

The generalized forces are a friction force

$$F_x = \lambda \frac{\partial f}{\partial x} = mg \sin\alpha \left(\frac{k_{cm}^2}{R^2 + k_{cm}^2} \right) \tag{119}$$

and a torque about the center of mass due to friction

$$F_\theta \equiv N = \lambda \frac{\partial f}{\partial \theta} = -mgR \sin\alpha \left(\frac{k_{cm}^2}{R^2 + k_{cm}^2} \right) \tag{120}$$

8 Generalized Forces That Do Work: Lagrange vs. Newton

In the previous section we discussed how to use the Lagrangian formulation to deal with situations with holonomic constraint forces. The constraint forces discussed did no work.

Section10.8 in the text discusses D'Alembert's Principle which allows Lagrangian techniques to be extended to a situation where some work is done by a non-conservative force such as friction. We will NOT go into any details, but only say that it is possible to re-formulate the Lagrangian approach to work here.

At the end of that section is a very important sub-section entitled ***Why the Lagrangian Method***. I will quote the key points, but encourage you to read that whole section.

> Lagrange's equations… provide an incredibly consistent methodology and indeed an almost mind-numingly mechanistic problem-solving strategy. … It seems to grant the practitioner omnipotent calculational prowess—enabling him or her to leap tall buildings with a single bound.…
>
> The strength of the Lagrangian approach to solving problems is based on its ability to deal with scalar functions, whereas the Newtonian approach is based on the use of the vector quantities, forces and momenta.
>
> When non-conservative or velocity-dependent generalized forces rear their ugly heads, however, the Newtonian method is invariably the one of choice; in such cases one really does need the sledgehammer to crack the walnut; delicate thrusts with the Lagrangian "rapier" will likely prove futile.

9 The Hamiltonian

For a system with N degrees of freedom, the Lagrangian method produces N second order differential equations of motion. An alternative approach, the Hamiltonian, produces $2N$

first order differential equations. The Hamiltonian really comes to the fore in Quantum Mechanics.

Define the Hamiltonian as

$$H = \sum_i \dot{q}_i \, p_i - L \tag{121}$$

When kinetic energy is an homogeneous quadratic function[6] of the generalized velocities, and the potential energy is a function of position, The Hamiltonian just becomes

$$H = T + V$$

Hamilton's Canonical Equations of Motion turn out to be

$$\frac{\partial H}{\partial p_1} = \dot{q}_i \tag{122}$$

$$\frac{\partial H}{\partial q_1} = -\dot{p}_i \tag{123}$$

i.e the aforementioned $2N$ first order differential equations.

Further discussion of the Hamiltonian will be deferred to the Quantum Mechanics course.

[6]Homogeneous functions of variables, signified as $\vec{v}$, have the property that scaling the variables by α scales the function by some power of α, or $f(\alpha\vec{v}) = \alpha^k f(\vec{v})$. For a quadratic function $\alpha = 2$.