This is an essay that I wrote a decade ago about Martin Gardner's Wallet Paradox, *which is also known as the* Envelope Paradox. *I'd known of this paradox—and I'd known its resolution—for many years when I noticed in 2002 that the paradox was getting some play in the American Mathematical Monthly and Mathematics Magazine. I wrote this paper in response to articles in those magazines. The editors asked for some revisions and…well, I never bothered with the revisions; writing the paper in the first place was something of a lark, and this bit of recreational math never again rose to the top of my to-do queue. That said, I do think that the logic of this paradox is valuable and interesting, so I thought I'd post this on my web page. --DSR, 2014.*

The Logic of the Wallet Paradox

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In a recent article [**1**] Merryfield et al. attempted to give a satisfactory explanation of the Wallet Paradox. This probabilistic paradox was introduced by Martin Gardner [**2**] as a variant of a paradox discussed by Kraitchik [**3**]. Here is the statement of the paradox given by Merryfield et al.:

> Each of two persons places his wallet on the table. Whoever has the smallest amount of money in his wallet wins all the money in the other wallet. Each of the players reasons as follows: "I may lose what I have but I may also win more than I have. So the game is to my advantage."

Kraitchik tried to resolve the paradox by considering a case in which each of the two persons carries some amount between \$0 and some specified amount, \$100, with all amounts being equally likely. He then constructed a payoff matrix and showed that the game is to no one's advantage.

Gardner pointed out that

> Unfortunately, this does not tell us what is wrong with the reasoning of the two players. We have been unable to find a way to make this clear in any simple manner. Kraitchik is no help, and so far as we know, there is no other reference on the game.

Merryfield et al. took up Gardner's challenge. They took the problem to be Kraitchik's use of a particular probability distribution; Kraitchik had given an example, not a proof. So, where Kraitchik considered only the uniform distribution on a finite interval, they considered a general class of probability densities and gave a nice proof that the game is to no one's advantage. They concluded their article by giving a deeper analysis of the effect on the game of the structure of the density functions, and by posing some questions about strategy in generalizations of the game. In [**4**], Carroll, Jones, and Rykken addressed these questions in game-theoretic terms. So, this renewed interest in the topic has been productive.

However, I do not think that Merryfield et al. have answered Gardner's question about what is wrong with the reasoning of the two players. I do not think that Gardner's objection to Kraitchik's discussion was that it was merely an example. I think that his objection was that while Kraitchik showed how to reason properly about the situation, he did not explain what was wrong with the fallacious reasoning. Merryfield et al. have shown more elaborately how to think of such probability problems correctly. But like Kraitchik, they have not said what is wrong with the original reasoning.

In this note, I will try to answer Gardner's question. First, I will present the same paradox in a different form. It is the form in which I first heard this paradox, and I think that it is a form that provides a slightly different, and valuable, perspective on the basic issue.

At the 1994 IMA Mathematical Modeling Workshop at the University of Minnesota, Professor Colin Please of the University of Southhampton presented me and others with the following paradox:

> Suppose that I choose a positive number at random and write a check for that amount. I put it in an envelope. I then write a check for double the amount, and put it in another, identical envelope. You choose an envelope, and you may keep and cash the check for the amount in the envelope. After you've chosen an envelope, but before you open it, I offer you the opportunity to switch envelopes. You reason as follows. You have, in your envelope, a check for an amount, call it X. If you switch, half the time you will trade down to X/2, half the time you will trade up to 2X. On the average, your expected value if you switch will be (X/2 + 2X)/2=1.25X, a 25% increase over the amount you have in hand. So of course, you'll switch. But then, you'll have in hand an amount Y, and again, your expected value for switching would be 1.25Y, so you'd switch. But then, … what's wrong?

This paradox has the same something-from-nothing quality that the Wallet Paradox has. I think that it also shares that paradox's underlying fallacy. I also think that the computation of an expected value, which is an important element of this paradox, draws attention to an important logical element—the role of an implicit probability distribution—that is more well-concealed in the Wallet Paradox.

The reasoning error in both the Wallet Paradox and the Envelope Paradox is the implicit assumption that there is a probability distribution that is "the uniform distribution on the positive real numbers." There is no such distribution.

In the Wallet Paradox, the players reason that independent of the amount one has in his wallet, it is equally likely that his amount is the larger amount or the smaller amount. This is based, I think, on the sort of implicit assumption that we often make in probability problems, that "all amounts are equally likely". This is not only untrue in practice—the probability that I have $25 in my wallet is considerably higher than the probability that I have $50000 there—but it is untrue in the world of mathematical idealizations; there is no such thing as a uniform probability distribution on the positive numbers.

In the Envelope Paradox, the use of this fallacious assumption is even clearer. The player reasons that, regardless of what X is, it is equally likely that the amount in the other envelope is X/2 and that it is 2X. This is true only if all positive numbers are equally likely to be the number written on the first check. But this is impossible because there is no uniform distribution on the positive real numbers.

There is, of course, a uniform distribution on any interval $[0, C]$. The density function for that distribution is simply the constant function $\rho_C(x) \equiv \frac{1}{C}, 0 \le x \le C$. However, this density has no sensible density as a limit as $C \to \infty$. In that limit, $\rho_C(x)$ simply approaches 0 uniformly, leaving nothing like a probability density; the limiting function cannot be normalized, and the moments of $\rho_C(x)$ all approach ∞.

Merryfield et al. have shown that if we assume that the amount of money one carries in one's wallet is given by a distribution that has a finite mean, then the game is not to anyone's advantage; one's expected gain in playing the game is 0. However, their reasoning does not apply to distributions that do not have finite means. Let us consider the example of a modified Cauchy distribution; it will give us a bit more insight into the nature of the paradox.

Our modified Cauchy distribution (it is modified in that it is defined only on the nonnegative real numbers, whereas the standard Cauchy distribution is defined on the whole line [**5**]), has the density $\rho(x) = \left(\frac{2}{\pi}\right)\frac{1}{1+x^2}$ and it is defined for $x \ge 0$. It is, as a proper density function must be, normalized, $\int_0^\infty \rho(x)dx = 1$. However, none of its moments are defined (or, to speak loosely, they are all infinite). For example, if we try to compute the mean of this distribution, we find

$$\int_0^\infty x\rho(x)dx = \lim_{t\to\infty}\int_0^t x\rho(x)dx = \frac{1}{\pi}\lim_{t\to\infty}\int_0^t \frac{2x}{1+x^2}dx = \frac{1}{\pi}\lim_{t\to\infty}\log(1+t^2) = \infty$$

Now suppose that I am a participant in the Wallet game, and that the contents of my wallet, and the contents of the other participant's wallet, are both given by the modified Cauchy distribution. Then if I have an amount x in my wallet, the probability that I will win the game is $\int_x^\infty \rho(\xi)d\xi$ and the probability that I will lose it is $\int_0^x \rho(\xi)d\xi$. When I lose, I lose an amount x, and when I win, I win, on average, an amount $\int_x^\infty \xi\rho(\xi)d\xi = \infty$. Clearly such a game would be to my advantage! The paradox is produced by the invalid probability density's making an infinitude of money available for the game.

The premise that the amounts in the wallets is satisfy a modified Cauchy distribution—a distribution with an infinite mean—has led to the absurd conclusion that my expected value for playing the game is infinite. But the assumption of a modified Cauchy distribution is a more sober, if less "intuitive" assumption than the assumption that one amount is as likely as any other. Both the "uniform distribution on the positive reals" and the modified Cauchy distribution have infinite means, but at least the Cauchy distribution can be normalized, whereas the "uniform distribution" cannot be normalized.

I am indebted to Jimmy Wales, who pointed out me, in the context of the envelope paradox, that a distribution with an undefined mean is sufficient to generate the paradox.

The best mathematical puzzles are those whose resolutions illuminate mathematical methods more brightly than simple exercises. A good paradox—a logic puzzle that asks us to resolve an apparent contradiction—serves a similar purpose. It makes us reflect on our methods of reasoning, and it helps us root out persuasive but subtly fallacious ways of thinking. When one is working on such puzzles and paradoxes, one is not, primarily, thinking about numbers and wallets and envelopes and probabilities, one is thinking about thinking. This is why I think that Gardner was right to press for an account of the error in reasoning behind the wallet paradox, why he was right not to be satisfied with an account of the proper way to reason about it.

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